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BLOW-UP POINTS OF SOLUTIONS TO ELLIPTIC EQUATIONS WITH LIMITING NONLINEARITY

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1. Introduction and results. Considering from a variational viewpoint the problem

$$\begin{cases} -\Delta u = u^p, \quad u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^N$, $N \geq 3$, and $p = \frac{N+2}{N-2}$, it appears that the functionals whose critical points are solutions to (P) do not satisfy the Palais-Smale condition — there exist critical points at infinity; that is, orbits along which the functional remains bounded and its gradient goes to zero, and which do not converge. For example, if we look at the functional

$$F(u) = \Big(\int_\Omega |u|^{p+1} \Big)^{-\frac{2}{p+1}} \int_\Omega |\nabla u|^2 \qquad \text{on } H_0^1(\Omega) - \{0\}$$

whose positive critical points are solutions to (P) up to a multiplicative constant, the Palais-Smale condition is violated for each level $k^{2/N} S_N$, $k \in \mathbb{N}^*$ ([2]) with S_N the Sobolev embedding constant, independent of Ω bounded in $\mathbb{R}^N$ ([3]),

$$S_N = \inf_{H_0^1(\Omega)-\{0\}} F(u) = \inf_{\substack{u \in L^p(\mathbb{R}^N)-\{0\} \\ \nabla u \in L^2(\mathbb{R}^N)}} \int_{\mathbb{R}^N} |\nabla u|^2.$$

Lack of compactedness at level $k^{2/N} S_N$ is due to blow-up phenomena which occur at k points (eventually identical) of sequences (u^n) in $H_0^1(\Omega)$,

$$|\nabla u^n|^2 \to S_N^{N/2} \sum_{i=1}^k \delta_{x_i}, \qquad |\nabla u^n|^{p+1} \to S_N^{N/2} \sum_{i=1}^k \delta_{x_i} \tag{1}$$

in the sense of measures, with δ_{x_i} the Dirac mass at $x_i \in \overline{\Omega}$.

We then have

$$u^n = \sum_{i=1}^k \alpha_i^n P\delta_{\lambda_i^n, x_i^n} + v^n \tag{2}$$

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