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NODAL SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS WITH CRITICAL EXPONENT†

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Abstract: Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with smooth boundary and $p = 2N/(N-2)$ be the critical Sobolev exponent. In this note we extend the results of [10] and [21] concerning nodal solutions (i.e., a solution which changes sign) for the Dirichlet problem: $-\Delta u = |u|^{p-2}u + \lambda u$ on Ω and $u = 0$ on $\partial\Omega$, when $N \geq 6$ and $\lambda \in (0, \lambda_1)$ with λ_1 the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. Similarly, for the problem $-\Delta u = |u|^{p-2}u + \lambda|u|^{q-2}u$ on Ω and $u = 0$ on $\partial\Omega$ we obtain a nodal solution when $\lambda > 0$, $(N+2)/(N-2) < q < 2N/(N-2)$ for $N = 3$, 4, 5 and $2 < q < 2N/(N-2)$ for $N \geq 6$.

Introduction and main results. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with smooth boundary $\partial\Omega$ and $N \geq 3$. Consider the Dirichlet problem:

$$-\Delta u = |u|^{p-2}u + \lambda u \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{*}$$

where $p = 2N/(N-2)$ is the best exponent in the Sobolev embedding and $\lambda > 0$. Define λ_1 to be the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. Using ideas introduced by Aubin ([1]) for the Yamabe problem, Brezis-Nirenberg ([6]) proved that in contrast to the non-existence situation yield by the Pohozaev's identity ([15]) for $\lambda \leq 0$, in case $N \geq 4$ and $\lambda \in (0, \lambda_1)$ then problem $(*)$ always admits a positive solution.

Notice that if $\lambda \geq \lambda_1$, every solution of $(*)$ must change sign in Ω. Existence in such situations has been established in [9]. The dimension $N = 3$ appears more delicate and existence is only possible when $\lambda > \lambda_* > 0$ for a suitable constant λ_* depending on the domain Ω (see [6]). We refer to [4] and [18] for a detailed bibliography related to various interesting aspects of this problem.

In this note, we will be concerned with solutions of $(*)$ which change sign in Ω. Following the notation introduced in [2], we shall refer to these solutions as nodal solutions. The existence of a pair of nodal solutions for $(*)$ has been obtained in [10] and [21] for $N \geq 6$ and $\lambda \in (0, \lambda_1)$.

Here we give a different proof of these results together with a mild extension. As in [10] and [21], we use variational methods. However, our proof relies more on the specific choice of the P.S. sequence then on the appropriate minimax principle. We hope that our point of view will shed some new light on the multiplicity question for problem $(*)$. We have:

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