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A GENERALIZED DERIVATIVE FOR CALM AND STABLE FUNCTIONS

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Abstract. A simple way of defining a generalized derivative for arbitrary functions is introduced. Its main feature lies in its coincidence with the usual derivative when the function is differentiable, a property which is not shared by the strict derivative of F. Clarke. Calculus rules are presented; in particular, functions satisfying a regularity condition weaker than a Lipschitzian property are shown to verify a chain rule. Applications to optimization are pointed out.

The first steps in nonsmooth analysis [20], [22], [26] relied on global or infinitesimal convexity assumptions. A breakthrough appeared with the introduction by F.H. Clarke [4-6], (see also [11-12], [27-31]) of a process which automatically incorporates convexity in the generalized derivatives. This enables one to dualize the notions; moreover the calculus rules satisfied by his generalized derivatives are quite workable. However, as observed by many authors who made alternative proposals [7-17], [21], [32-35], the optimality condition involving the Clarke's subdifferential

$$0 \in \partial^0 f(x)$$

generalizing the classical condition $f'(x) = 0$ is rather loose and so are the necessary conditions one can deduce from it in various fields as mathematical programming or optimal control theory. Moreover, for a differentiable function its Clarke subdifferential $\partial^0 f(x)$ may differ from $\{f'(x)\}$. This stems from the fact that the Clarke derivative is not a generalization of the derivative but a generalization of the *strict derivative*

$$\lim_{(t,w)\to(0_+,x)} t^{-1}(f(w+ty) - f(w))$$

used in [2], [3]. Accordingly the generalized strict derivative of Clarke has achieved important success in generalizing the inverse function theorem (and related results) in which the strict derivative plays a key role in the smooth case.

The purpose of the present paper is to introduce some simple ways of defining a generalized derivative which is convex but close enough to the usual derivative when it exists. This is done by retaining some part of the notion of strict differential.

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