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BEHAVIOR OF DIRECTIONS OF SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Abstract. Let K be a cone with nonempty interior $\dot{K}$ in a Banach space X and $f : \mathbb{R}_+ \times K \to X$ be positively homogeneous of degree 1. Denote by $x(t,\xi)$ the solution of $x' = f(t,x)$ such that $x(0,\xi) = \xi$. In this paper we are interested in what further conditions on f imply that for any $\xi_0, \xi_1 \in \dot{K}$ one has $\|\frac{x(t,\xi_0)}{\|x(t,\xi_0)\|} - \frac{x(t,\xi_1)}{\|x(t,\xi_1)\|}\| \to 0$ as $t \to \infty$. The results are obtained via Hilbert's projective metric. These results are applied to the case when K is a standard cone in $\mathbb{R}^n$ and f belongs to the class M which arises in some models from population biology.

Introduction. Let K be a cone with nonempty interior $\dot{K}$ in a Banach space X. Let $f : \mathbb{R}_+ \times K \to X$ and $x(t,\xi)$ denote a solution of $x' = f(t,x)$ such that $x(0) = \xi$. By a direction of the solution $x(t,\xi)$ we will mean a vector $\frac{x(t,\xi)}{\|x(t,\xi)\|}$. We will be interested in the bevavior of $\frac{x(t,\xi)}{\|x(t,\xi)\|}$ as $t \to \infty$. In general one cannot expect that directions converge to some point of the cone but one can ask a question when for any two points $\xi_0, \xi_1 \in \dot{K}$ the directions $\frac{x(t,\xi_0)}{\|x(t,\xi_0)\|}$ and $\frac{x(t,\xi_1)}{\|x(t,\xi_1)\|}$behave in a similar way. For example, what conditions on f imply that

$$\|\frac{x(t,\xi_0)}{\|x(t,\xi_0)\|} - \frac{x(t,\xi_1)}{\|x(t,\xi_1)\|}\| \to 0 \quad \text{for all } \xi_1, \xi_1 \in \dot{K}.$$

If $f(t,\cdot) = A(t)$ and $A(t)$ is an essentially nonnegative matrix this question was treated in [1], [2] (see also [13]). Birkhoff and Kotlin proved that if there are irreducible matrices B and C such that $B \leq A(t) \leq C$ then the solutions are eventually positive and $\|\frac{x(t)}{\|x(t)\|} - \frac{y(t)}{\|y(t)\|}\| \to 0$ for any solutions $x(t), y(t)$. Our motivation for studing this problem comes from some models from population biology. In population biology literature such results are called "weak ergodic theorems".

We would like to mention that there is an enormous amount of work in the discrete version of this problem. For the case of linear operators in finite dimensional spaces, the asymptotic behavior of products of matrices $A_n A_{n-1} \cdots A_n$ have been investigated in the theory of Markov chains and in the theory of nonnegative matrices. Recently Nussbaum (see [9] and references therein) investigated the behavior of F_n where $F_n = f_n \circ f_{n-1} \cdots \circ f_1$ and $f_i : K \to K$ are homogeneous of order 1 and order-preserving.

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