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DIFFERENTIABILITY OF THE OPTIMAL COST FUNCTION IN POINTWISE CONTROL

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Abstract. In this paper, we study the pointwise control for the heat equation $y_t + Ay = v\delta(x - \mathbf{b})$ with homogeneous initial data and boundary conditions. Here, $\delta(x - \mathbf{b})$ is the Dirac mass at $\mathbf{b}$. The cost function is given by $J(v) = \int_\Omega |y(T;v) - z_d|^2\,dx + N\int_0^T |v|^2\,dt$. The optimal cost function $j(\mathbf{b})$ is defined as the value of J at the optimal control $u(= u(\mathbf{b},\cdot))$ which is the solution of $J(u) = \inf\{J(v) : v \in U_{ad}\}$, where U_{ad} is a closed convex subset of the control space U. We prove that j is differentiable with continuous partial derivatives for any closed convex subset U_{ad} of U for which the projection operator $\mathcal{P}$ of U on U_{ad} satisfies $\mathcal{P}(v(t)) = f(v(t))$, in some neighbourhood of T, for some nondecreasing continuous real function f.

1. Introduction. In this paper, we will study the regularity of the *optimal cost function*. This problem appears in pointwise control of systems governed by parabolic partial differential equations that typically arise in heat conduction or diffusion problems. In order to fix ideas, let us consider the temperature in a homogeneous body Ω that is controlled by a variable heat source $v(t)$ applied at the point $b \in \Omega$, $t > 0$. If we denote the temperature corresponding to v by $y(t, x : v)$ (or just $y(t : v)$ or $y(v)$) and if the boundary Γ of $\Omega \subset \mathbb{R}^n$ is kept at a constant, say zero, temperature, then y satisfies

$$\begin{aligned} &y_t + Ay = v\delta(\mathbf{x} - \mathbf{b}) && \text{in } Q = \Omega\times]0,T[,\\ &y(0,\mathbf{x}) = 0 && \text{in } \Omega, \\ &y(t,\mathbf{x}) = 0 && \text{on } \Sigma = \Gamma\times]0,T[, \end{aligned} \tag{1.1}$$

where for convenience we are assuming that there is not another external heat source and the initial temperature of the body is zero. Here $\delta(\mathbf{x} - \mathbf{b})$ is the Dirac mass at $\mathbf{b}$ and A is a symmetric second order differential operator of the form

$$A = -\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial}{\partial x_j}\left\{a_{ij}\frac{\partial}{\partial x_j}\right\} + c_0 \tag{1.2}$$

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