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A NOTE ON A CONTINUATION PRINCIPLE FOR COMPACT PERTURBATIONS OF THE IDENTITY

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Abstract. The equation $F(x,\lambda) = x - k(x,\lambda)$, $(x,\lambda) \in \mathcal{O} \subseteq \mathcal{X} \times \mathbb{R}^n$, $F(0,0) = 0$, where $\mathcal{X}$ is a real Banach space, is considered. The nonlinear operator k is assumed to be continuous on $\mathcal{O}$ and compact on the open sets $\mathcal{O}(\epsilon)$, where $\mathcal{O} = \cup_{0<\epsilon<E}\mathcal{O}(\epsilon)$. F is differentiable at $(0,0)$ and $\ker DF(0,0)$ has dimension n. It is shown by means of the Leray-Schauder degree that the connected component of the set of zeros of F containing $(0,0)$ is either unbounded, approaches $\partial\mathcal{O}$ in a well-defined sense or intersects all the subspaces $\mathcal{Y}$ of condimension n in $\mathcal{X} \times \mathbb{R}^n$ such that $\ker DF(0,0) \cap \mathcal{Y} = \{(0,0)\}$ at a point distinct from $(0,0)$. The result is known if the Leray-Schauder topological degree of the map $F(\,\cdot\,,0)$ relative to some open and bounded subset of $\mathcal{O} \cap (\mathcal{X} \times \{0\})$ containing $(0,0)$ is defined and different from $0, \mathcal{Y} = \mathcal{X} \times \{0\}$, and k is compact on $\mathcal{O}$. References for applications are given.

1. Introduction. The purpose of the present note is to prove a consequence of a continuation principle given in J. Ize, I. Massabò, J. Pejsachowicz, and A. Vignoli [2, Theorem 4.1]. Our proof is self-containing. We consider the (celebrated) equation

$$F(x,\lambda) = 0, \quad F(x,\lambda) \equiv x - k(x,\lambda), \quad k(0,0) = 0, \quad (x,\lambda) \in \mathcal{O} \subseteq \mathcal{X} \times \mathbb{R}^n, \quad (1.1)$$

where k is continuous from the open subset $\mathcal{O}$ containing $(0,0)$ to the real Banach space $\mathcal{X}$. We assume that $\mathcal{O}$ is the union of an ascending family of open sets $\mathcal{O}(\epsilon)$ (cf. (2.5)) and that k is compact on each $\mathcal{O}(\epsilon)$, although k is not assumed to be compact on $\mathcal{O}$. This situation seems to occur a number of times in the study of fluid-solid interaction problems (cf. [3–5], [6]), where this paper finds application. We now briefly summarize our statement. Let $\mathcal{S}_0$ be the connected component of the set of zeros of F in $\mathcal{O}$ containing $(0,0)$. Let F be differentiable at $(0,0)$ and let $\ker DF(0,0)$ denote the null space of $DF(0,0)$, which we assume to be of dimension n. Then either $\mathcal{S}_0$ is unbounded, approaches $\partial\mathcal{O}$ in the sense that $\mathcal{S}_0$ cannot be contained in any of the $\mathcal{O}(\epsilon)$ or $\mathcal{S}_0 \setminus \{(0,0)\}$ intersects every subspace $\mathcal{Y}$ of $\mathcal{X} \times \mathbb{R}^n$ of codimension n with $\ker DF(0,0) \cap \mathcal{Y} = \{(0,0)\}$. This conclusion is known to hold if k is compact on $\mathcal{O}$, $\mathcal{Y} = \mathcal{X} \times \{0\}$ and if the Leray-Schauder topological degree of the map $F(\,\cdot\,,0)$ relative to some open and bounded subset of $\mathcal{O} \cap (\mathcal{X} \times \{0\})$ containing $(0,0)$ is defined and different from 0 (cf. J. Ize, I. Massabò, J. Pejsachowicz and A. Vignoli [2, Theorem 4.1]).

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