Differential and Integral Equations, Volume 5, Number 4, July 1992, pp. 793–804.

FORCED SECONDARY BIFURCATION IN AN ELLIPTIC BOUNDARY VALUE PROBLEM

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(Submitted by: Peter Hess)

Abstract. A semi-linear elliptic boundary value problem with cubic nonlinearity is studied. In a previous paper the author has shown that for certain parameter values the equation under consideration has, for arbitrary forcing terms, at most three solutions. Here it is shown that for other parameter values a *secondary bifurcation* occurs. In particular, it is shown that for certain forcing terms the equation has at least five solutions. This result is obtained by studying the local structure of the associated nonlinear operator using arguments of singularity theory.

1. Introduction. In [10] the following equation

$$\begin{cases} -\Delta u - \lambda u + u^3 = h, & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0, & \text{on } \partial\Omega \end{cases} \tag{1}$$

was studied. Here $\Omega \subset \mathbb{R}^n$ is a bounded and smooth domain, $\lambda \in \mathbb{R}$ is a parameter and $h \in C^\alpha(\Omega)$, with $\alpha \in (0,1)$ fixed, is a given forcing term.

The approach taken in [10] was to consider the corresponding nonlinear operator

$$\Phi = \Phi(\lambda) := -\Delta - \lambda + (\cdot)^3 : E \to F,$$

where $E = \{u \in C^{2,\alpha}(\Omega) : \frac{\partial u}{\partial n}\,|_{\partial\Omega} = 0\}$ and $F = C^\alpha(\Omega)$, and to apply *singularity theory in Banach space* to study the geometry of the mapping Φ. This method has been proposed first by Ambrosetti-Prodi in their influential paper [1].

We denote by $S = S(\lambda) = \{u \in E : \exists v \in E \setminus \{0\} \text{ with } \Phi'(u)[v] = 0\}$ the *singular set of* Φ (here Φ' denotes the Fréchet derivative of Φ in E). Let $0 = \lambda_1 < \lambda_2 \le \lambda_3 \le \dots$ denote the eigenvalues of the Laplacian on Ω with Neumann boundary conditions.

It was shown in [10] that for $0 = \lambda_1 < \lambda < \lambda_2$ the set S is a starshaped smooth manifold of codimension one in the Banach space E (see also Church-Timourian [6]). To study the image of the singular set under Φ (and thus obtain information on the solution structure of (1)) one needs a classification of the singular points. It was proved in [10] that for $0 < \lambda < \frac{1}{7}\lambda_2$ the singular set S consists entirely of *fold points* and *cusp points*, the first two singularities in the classification of R. Thom (see also Church-Dancer-Timourian [7] for the Dirichlet case). Based on these results,

Received July 1991, in revised form October 1991.
AMS Subject Classification: 35J65, 35B32.