Differential and Integral Equations, Volume 5, Number 6, November 1992, pp. 1363–1381.

ON THE STRUCTURE OF SEPARATRIX-SWEPT REGIONS IN SINGULARLY-PERTURBED HAMILTONIAN SYSTEMS

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(Submitted by: H.B. Keller)

Abstract. In this paper, we study the invariant structures in separatrix-swept regions of adiabatic (singularly-perturbed) planar Hamiltonian systems. Our main result is that lobe area is $\mathcal{O}(1)$ asymptotically. This result has important consequences for determining the structure in the region complementary to which Kruskal's adiabatic invariance theory applies, for transport in many applications, and for attempts at proving the existence of an $\mathcal{O}(1)$–sized stochastic or chaotic region in applications. We use adiabatic Melnikov perturbation theory to approximate an exact action-theoretic result which is a generalization of results in [16].

1. Introduction. Various physical, chemical, and fluid mechanical processes are modeled mathematically by planar Hamiltonian systems in which the Hamiltonian depends on a slowly varying parameter z; see [1, 3–5, 20, 22, 26] and especially the references in [3] and [5] for examples. The equations of motion are

$$\dot{q} = \frac{\partial H}{\partial p}(p,q,z), \quad \dot{p} = -\frac{\partial H}{\partial q}(p,q,z), \quad \dot{z} = \epsilon, \tag{1.1}$$

where $p, q, z \in \mathbb{R}$ and $0 < \epsilon \ll 1$. In these systems, the theory of adiabatic invariance (see [14]) and the extension of the KAM Theorem to adiabatic systems (see [1]) describe the dynamics of all orbits which lie sufficiently far from instantaneous separatrices (zero frequency motions). However, in many problems of interest, instantaneous separatrices sweep out large $\mathcal{O}(1)$ regions and, as a result, large classes of orbits are forced to cross instantaneous separatrices, changing orbit-type and their adiabatic invariant as they do so. For these orbits, the separation in scales between the $\mathcal{O}(\epsilon)$–rate of slow modulation and the instantaneous orbital frequency, which is a fundamental hypothesis in the above theories, ceases to exist as a separatrix is approached because the orbital frequency vanishes; see [4].

There are two types of results, which are distinct yet complementary, one can obtain about separatrix-swept regions. First, one can seek representations for individual separatrix-crossing orbits. This goal has been pursued successfully using asymptotic matching analysis in action-angle coordinates, which is referred to as separatrix-crossing theory; see for example [4], [5], [20], or the concise review in [3]. Second, one can try to determine the geometry of the invariant structures which

Received May 1991, in revised form February 1992.
AMS Subject Classifications: 58F30, 34C35, 34D15, 34A34.