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PONTRYAGIN MAXIMUM PRINCIPLE FOR SEMILINEAR SECOND ORDER ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS AND VARIATIONAL INEQUALITIES WITH STATE CONSTRAINTS

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Abstract. We present a method for deriving first order necessary conditions of optimal controls for semilinear second order elliptic partial differential equations and variational inequalities with various kinds of state constraints. The main tools used are the Ekeland variational principle and the spike variation technique for elliptic equations which is a combination of the ideas from vector valued measure theory and the representation of solutions for second order partial differential equations via Green's functions.

1. Introduction. The purpose of this paper is to present a method for deriving a Pontryagin type maximum principle as a first order necessary condition of optimal controls for problems governed by semilinear elliptic partial differential equations and variational inequalities. We allow various kinds of constraints to be imposed on the state. To be more precise, let us take the case of semilinear variational inequalities as an example. Thus, we have the system

$$\begin{cases} Ay(x) + \beta(y(x)) \ni f(x, y(x), u(x)) & \text{in } \mathcal{D}'(\Omega), \\ y \mid_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where Ω is a bounded region in $\mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$, A is the second order elliptic partial differential operator

$$Ay(x) = -\sum_{i,j=1}^{n} \partial_{x_i}(a_{ij}(x)\partial_{x_j} y(x)), \tag{1.2}$$

$\beta \subset \mathbb{R} \times \mathbb{R}$ is a maximal monotone graph with $0 \in \mathrm{Dom}\,(\beta)$, $f : \Omega \times \mathbb{R} \times U \to \mathbb{R}$ is a given map and U is a metric space in which the control variable $u(\cdot)$ takes values. Under certain conditions, which will be specified later, for each $u(\cdot) \in \mathcal{U} \equiv \{u : \Omega \to U \mid u(\cdot) \text{ measurable}\}$, there exists a $y(\cdot)$ in some function space $\mathcal{Y}$ satisfying (1.1) in a suitable sense. We refer to such a $y(\cdot)$ as a state associated with the control $u(\cdot)$. Then, we can talk about the state constraint

$$y(\cdot) \in Q, \tag{1.3}$$

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