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ON NONUNIQUENESS OF VISCOSITY SOLUTIONS

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Introduction. We look at the "Dirichlet problem", hereafter denoted by (DP), consisting of the "linear" ordinary differential equation

$$\mathcal{L}u = -a(x)u'' + c(x)u = f(x) \quad \text{on } (-1,1) \tag{1}$$

and the boundary conditions

$$u(-1) = \gamma_-, \qquad u(1) = \gamma_+. \tag{2}$$

We will *always* assume that

$$a, c, f \in C([-1,1]), \quad a, c \geq 0, \quad \gamma_-, \gamma_+ \in \mathbb{R}. \tag{3}$$

At issue is the existence and uniqueness of viscosity solutions of (DP). The notion of a viscosity solution of (DP), which is reviewed in Section 1, is appropriate for the study of certain fully nonlinear partial differential equations of second order. The reader may find a good recent account of the (now highly developed) general theory in [1]. In the context of this general theory (DP) may at first appear foolishly simple, but we are interested in refining our understanding of the interaction of existence and uniqueness questions with structure conditions on the equation under consideration. Clearly one should see what light (DP) might cast on this question, and it turns out that even this "simple" problem exhibits interesting subtleties. These subtleties can only arise if (DP) has singularities, that is, if the zero set of a

$$Z(a) = \{x \in [-1,1] : a(x) = 0\}$$

is nonempty, so we treat cases for which this is so. In order that the discussion does not become too ragged, we also usually assume that the "boundary" $\{-1,1\}$ is not singular. That is, we assume

$$-1, 1 \notin Z(a). \tag{4}$$

One of the main results of our study is the following theorem. In the statement "*solution*" is understood to mean "*viscosity solution*"; this convention is used throughout this work. Both assertions of the theorem are false if one uses most other notions of solution. The definition of a viscosity solution of (DP) is recalled in Section 1.

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