Differential and Integral Equations, Volume 5, Number 5, September 1992, pp. 1111–1114.

AN EXAMPLE OF A BLOW-UP SEQUENCE FOR $-\Delta u = V(x)e^u$

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(Submitted by: Haim Brezis)

1. Introduction. In this paper we consider the equation

$$\begin{cases} -\Delta u = V(x)e^u & \text{in } \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain and $V(x)$ is a given function in $L^\infty(\Omega)$.

In [1], H. Brezis and F. Merle study this equation and obtain the following uniform estimates for the solutions of (1).

Theorem 1. *Assume (u_n) is a sequence of solutions of (1) which satisfies*

$$\|V_n(x)\|_{L^\infty} < C, \tag{2}$$

$$V_n(x) \geq 0 \quad \text{on } \Omega, \tag{3}$$

and

$$\|e^{u_n}\|_{L^1} \leq C. \tag{4}$$

Then (u_n) is bounded in $L^\infty_{\text{loc}}(\Omega)$.

The aim of this paper is to answer a question raised by H. Brezis and F. Merle. Namely, we show that condition (3) is essential for this theorem.

2. Construction of the sequences (u_n) and (V_n). We shall prove the following theorem in this section:

Theorem 2. *There exist sequences (V_n) and (u_n) in $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$ satisfying (2) and (4) (but not (3)) with $u_n \geq 0$ in Ω such that $u_n(0) \to +\infty$.*

For $t > 0$, let D_t be the domain $D_t = \{x \in \mathbb{R}^2 : |x| < t\}$ and for $1 \geq a \geq b > 0$, let $\sigma_{a,b}(x)$ be the function defined on D_1 by

$$\sigma_{a,b}(x) = \begin{cases} 1 & \text{on } D_a \setminus D_b, \\ 0 & \text{otherwise}. \end{cases}$$

Let $u = u_{a,b}$ be the solution of

$$\begin{cases} -\Delta u = -\sigma_{a,b}e^u + \frac{14}{3}\pi\delta & \text{in } D_1, \\ u = 0 & \text{on } \partial D_1, \end{cases} \tag{5}$$

where δ is the Dirac function at $x = 0$.

It is well-known that (5) admits an unique solution. One may, for example, use sub and super solutions to obtain the existence and use the maximum principle to obtain the uniqueness.

Received for publication April 1991.