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ON SEMILINEAR EVOLUTION EQUATIONS WITH MANY LYAPUNOV FUNCTIONALS

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Abstract. A technique is described for proving time-asymptotic estimates in strong topologies for solutions of semilinear evolution equations from such estimates for weak Lyapunov functionals. Applications include parabolic equations with nonlinear transport terms, weakly coupled parabolic systems, and a class of fourth order parabolic equations.

0. Introduction. For many partial differential equations of evolution type, functionals of the solution exist whose behavior can be controlled as a function of time, either by differentiating them along a solution and using the differential equation or by converting the differential equation into some equivalent integral equation and estimating. Depending on the functional, this gives useful information on the local or global behavior of the solution. If there is a choice of such functionals, each will usually lead to a different estimate. In addition, it can happen that for some functionals global stability can be proved and for other ones only local stability. For the purpose of motivation, we discuss the semilinear heat equation

$$u_t = u_{xx} + f(u) \quad (x \in \Omega = (0,\pi) \subset \mathbb{R},\ t > 0) \tag{0.1}$$

with zero boundary data at $x = 0, \pi$ and suitable initial data on $[0,\pi] \times \{0\}$. We assume that $f\colon \mathbb{R} \to \mathbb{R}$ is smooth and that for some $C > 0$ and for all $u \in \mathbb{R}$,

$$uf(u) \le 0, \quad |f'(u)| \le C|u|^2. \tag{0.2}$$

Then any L^p-norm $\|\cdot\|_p$ of the solution can be estimated in the following standard fashion for $1 < p < \infty$ (the t-arguments are omitted):

$$\begin{aligned}\frac{d}{dt}\|u\|_p &= \|u\|_p^{1-p}\int_\Omega |u|^{p-2}uu_t = \|u\|_p^{1-p}\int_\Omega (u_{xx}+f(u))|u|^{p-2}u \\ &\le \|u\|_p^{1-p}(1-p)\int_\Omega |u|^{p-2}|u_x|^2 = \|u\|_p^{1-p}\frac{4(1-p)}{p^2}\int_\Omega |(u^{p/2})_x|^2.\end{aligned}$$

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