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LOG-CONCAVITY OF THE PRINCIPAL EIGENFUNCTION OF A LINEAR PERIODIC-PARABOLIC EIGENVALUE PROBLEM

M. Bitterlin
Institute of Mathematics, Universität Zürich, Rämistrasse 74, 8001 Zürich, Switzerland

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Abstract. A famous result by Brascamp-Lieb [2] says that in a bounded strongly convex domain, the principal eigenfunction of $-\Delta$, subject to Dirichlet boundary conditions, is log-concave. In this paper we prove, the same result for a periodic-parabolic eigenvalue problem. This result does not immediately follow from the known concavity results (e.g., [4]) since it is always assumed there that one starts with a log-concave initial condition (whereas this is one of the unknowns in periodic problems).

1. Introduction. Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded domain with boundary of class $C^{2+\mu}$, $\mu \in (0,1)$. In this note, we study the log-concavity of the principal eigenfunction of the linear periodic-parabolic eigenvalue problem

$$\begin{cases} \partial_t u - \sum_{j,k=1}^{N} a_{jk}(t)\partial_j\partial_k u + \sum_{j=1}^{N} a_j(t)\partial_j u = \lambda m(x,t)u & \text{in } \Omega \times \mathbb{R}, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}, \\ u(x,t+T) = u(x,t) & \text{in } \Omega \times \mathbb{R}, \end{cases} \tag{1}$$

where $T > 0$ is a given period and the weight function m belongs to the real Banach space

$$E := \{w \in C^{\mu,\frac{\mu}{2}}(\bar{\Omega} \times \mathbb{R}) : \ w \text{ is } T\text{-periodic in } t\ \}.$$

The T-periodic coefficient functions $a_{jk} = a_{kj}, a_j$ belong to to the class $C^{\frac{\mu}{2}}$ and the matrix $[a_{jk}(t)]$ is positive definite for all $t \in \mathbb{R}$.

By a result of Beltramo and Hess [1], problem (1) has a positive eigenvalue $\lambda_1(m)$ having a positive eigenfunction Φ lying in the real Banach space

$$F := \{w \in C^{2+\mu,1+\frac{\mu}{2}}(\bar{\Omega} \times \mathbb{R}) : w = 0 \text{ on } \partial\Omega \text{ and } w \text{ is } T\text{-periodic in t}\ \}$$

if and only if $\int_0^T \max_{x\in\bar{\Omega}} m(x,t)dt > 0$. Moreover, $\lambda_1(m)$ is the unique positive eigenvalue having a positive eigenfunction Φ and it is simple. The notions of positive principal eigenvalue and principal eigenfunction are thus justified.

The aim of this note is to prove the following:

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