Differential and Integral Equations, Volume 5, Number 5, September 1992, pp. 973–997.

FLAT BLOW-UP IN ONE-DIMENSIONAL SEMILINEAR HEAT EQUATIONS

M.A. HERRERO AND J.J.L. VELÁZQUEZ*
Departamento de Matemática Aplicada, Facultad de Matemáticas
Universidad Complutense, 28040 Madrid, Spain

(Submitted by: H. Brezis)

Abstract. Consider the Cauchy problem

$$u_t = u_{xx} + u^p, \quad x \in \mathbb{R}, \quad t > 0,$$
$$u(x,0) = u_0(x), \quad x \in \mathbb{R},$$

where $p > 1$ and $u_0(x)$ is continuous, nonnegative and bounded. Assume that $u(x,t)$ blows up at $x = 0$, $t = T$ and set

$$u(x,t) = (T-t)^{-\frac{1}{p-1}} \phi(y,\tau), \; y = \frac{x}{\sqrt{T-t}}, \; \tau = -\ln(T-t).$$

Here we show that there exist initial values $u_0(x)$ for which the corresponding solution is such that two maxima collapse at $x = 0$, $t = T$. One then has that

$$\begin{aligned} \phi(y,\tau) =& (p-1)^{\frac{1}{p-1}} - C_1 e^{-\tau} H_4(y) + o(e^{-\tau}) \quad \text{as } \tau \to \infty, \\ & \text{with } C_1 > 0, \; H_4(y) = c_4 \tilde{H}_4(y/2), \end{aligned} \tag{1}$$

where $c_4 = (2^3(4\pi)^{1/4})^{-1}$, $\tilde{H}_4(s)$ is the standard 4^{th}-Hermite polynomial, and convergence in (1) takes place in $C^{k,\alpha}_{\text{loc}}$ for any $k \geq 1$ and some $\alpha \in (0,1)$. We also show that in this case,

$$\lim_{t \uparrow T} (T-t)^{\frac{1}{p-1}} u(\xi(T-t)^{1/4}, t) = (p-1)(1 + C_1 c_4 \xi^n)^{-\frac{1}{p-1}}, \tag{2}$$

where the convergence is uniform on sets $|\xi| \leq R$ with $R > 0$. This asymptotic behaviour is different (and flatter) than that corresponding to solutions spreading from data $u_0(x)$ having a single maximum, in which case

$$\phi(y,\tau) = (p-1)^{-\frac{1}{p-1}} - \frac{(4\pi)^{1/4}(p-1)^{-\frac{1}{p-1}}}{\sqrt{2}\,p} \cdot \frac{H_2(y)}{\tau} + o(\frac{1}{\tau}) \text{ as } \tau \to \infty, \tag{3}$$

$$\lim_{t \uparrow T} (T-t)^{\frac{1}{p-1}} u(\xi(T-t)^{1/2} |\ln(T-t)|^{1/2}, t) = (p-1)^{-\frac{1}{p-1}} (1 + \frac{(p-1)}{4p} \xi^2)^{-\frac{1}{p-1}}. \tag{4}$$

1. Introduction and description of results.

Here we consider the following Cauchy problem

$$u_t = u_{xx} + u^p \quad \text{when } -\infty < x < +\infty, \quad t > 0, \tag{1.1}$$
$$u(x,0) = u_0(x) \quad \text{when } -\infty < x < +\infty, \tag{1.2}$$

Received for publication September 1991.
*Partially supported by CICYT research grant PB90-0235 and EEC contract SC1-0019-C
AMS Subject Classification: 35B40, 35K55, 35K57.