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RATIONAL SOLUTIONS OF THE FIFTH PAINLEVÉ EQUATION

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Abstract. Consider the fifth Painlevé equation:

$$(P_5) \qquad y'' = (\frac{1}{2y} + \frac{1}{y-1})\,(y')^2 - \frac{y'}{z} + \frac{(y-1)^2}{z^2}\,(\alpha y + \frac{\beta}{y}) + \frac{\gamma y}{z} + \frac{\delta y\,(y+1)}{y-1}$$

where $\alpha, \beta, \gamma, \delta$ are complex parameters. Following Murata's systematic method, we prove the necessary and sufficient conditions for the existence of rational solutions of this equation in terms of special relations among the parameters. The number of rational solutions in each case is also proved and each of them may be generated recursively by means of Bäcklund transformations found by Gromak. A list of rational solutions is included in the appendix. They are computed using Mathematica. These rational solutions imply exact solutions for some associated evolution equations.

1. Introduction. At the turn of the century, Painlevé and his school [33, 9] classified all differential equations of the form $y'' = F(z, y, y')$ on the complex plane where F is rational in y and y', locally analytic in z, and for each solution, all the singularities which are dependent on the initial conditions have to be poles. They found 50 equations of this type, of which 44 can be integrated in terms of classical functions and transcendents, elliptic functions or transformed into the remaining six equations. These six equations are called the *Painlevé equations* (P_1–P_6). Below we list the second and fifth Painlevé equations, where $\alpha, \beta, \gamma, \delta$ are complex parameters:

$$(P_2) \qquad y'' = z\,y + 2\,y^3 + \alpha,$$

$$(P_5) \qquad y'' = (\frac{1}{2y} + \frac{1}{y-1})\,(y')^2 - \frac{y'}{z} + \frac{(y-1)^2}{z^2}\,(\alpha y + \frac{\beta}{y}) + \frac{\gamma y}{z} + \frac{\delta y\,(y+1)}{y-1}.$$

Recently, it has been discovered that the Painlevé equations are related to a wide range of physical problems such as statistical mechanics and quantum field theory (see reference list in [16]). A remarkable example is that the 2-point correlation

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