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THE NAVIER-STOKES EQUATION FOR AN INCOMPRESSIBLE FLUID IN $\mathbb{R}^2$ WITH A MEASURE AS THE INITIAL VORTICITY

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Dedicated to the memory of Peter Hess

Abstract. Global (in time) solutions of the Navier-Stokes equation, and the associated vorticity equation, for an incompressible fluid in $\mathbb{R}^2$ are constructed, with a measure ω as the initial vorticity. Regularity of the velocity and the vorticity fields as well as monotonicity (in time) of the L^p-norms of the vorticity are proved. Estimates for the singularity at $t = 0$ and the decay rate at $t = \infty$ of their L^p-norms are deduced, and found to be almost identical with those for the solutions of the linear (heat) equation. Uniqueness is proved under a mild restriction on the atomic part ω_a of ω, with no restriction on the size of the continuous part ω_c. For example, it suffices that the ℓ^p-norm of the atoms (regarded as a sequence) for some $p \in [4/3, 2)$ do not exceed a certain numerical value η_p, which is explicitly given.

Introduction. This paper is concerned with the initial value problem for the Navier-Stokes equation for an incompressible fluid in $\mathbb{R}^2$, and the associated vorticity equation. The former may be written in the form (cf. [11,12]):

$$\text{(NS)} \qquad \partial_t u - \Delta u + \Pi\partial(u \otimes u) = 0, \quad u = \Pi u, \quad (\partial_t = \partial/\partial t, \ \partial = \text{grad}),$$

where $u = u(t, x)$ is the velocity field, Π is the projection onto solenoidal vectors along gradients, $u \otimes u$ is a tensor with jk-component $u_k u_j$, and $\partial(u \otimes u)$ is a vector with j-th component $\partial_k(u_k u_j) = u_k \partial_k u_j$ (summation convention). The kinematic viscosity is set equal to one.

The associated (scalar) vorticity $\zeta = \partial \wedge u = \partial_1 u_2 - \partial_2 u_1$ satisfies the vorticity equation

$$\text{(VOR)} \qquad \partial_t \zeta - \Delta\zeta + \partial \cdot (\zeta S * \zeta) = 0, \quad S(x) = (2\pi)^{-1}|x|^{-2}(x_2, -x_1),$$

where $*$ denotes convolution. $S*$ is a linear operator such that $u = S * \zeta$ solves the equations $\partial \cdot u = 0$ and $\partial \wedge u = \zeta$, and has the continuity property (Hardy-Littlewood-Sobolev inequality)

$$\|S * \phi\|_p \leq \sigma_q \|\phi\|_q \quad \text{for } 1/p = 1/q - 1/2, \quad 1 < q < 2, \tag{0.1}$$

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