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THE PERRON INTEGRAL IN ORDINARY DIFFERENTIAL EQUATIONS*

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Abstract. The integral form of the initial value problem $\dot{x} = f(t,x), \quad x(\alpha) = v$ for an ordinary differential equation is $x(t) = x(\alpha) + \int_\alpha^t f(s,x(s))\,ds$. Results are obtained when the integral in this equation is treated as the Perron integral. The Henstock–Kurzweil summation approach to the Perron integral is used extensively. It is shown that all known conditions for the existence of a solution concern the case of a Carathéodory right hand side perturbed by a Perron integrable function.

The present approach to the concept of an ordinary differential equation goes back to C. Carathéodory, in particular to his book [3] published in 1918. In this work Carathéodory accomplished the construction of a calculus course based purely on the concept of the Lebesgue integral.

To solve an ordinary differential equation of the form

$$\dot{x} = f(t,x), \tag{1}$$

with $f : [a,b] \times B \to \mathbb{R}^n$ where $B \subset \mathbb{R}^n$ is an open set (e.g., $B = B_c = \{x \in \mathbb{R}^n : \|x\| \le c\}$), in the classical setting means:

Find (if possible, all) functions $x : J \to \mathbb{R}^n$ defined on a nondegenerate interval $J \subset [a,b]$ such that

$$x(t) \in B \textit{ for } t \in J, \tag{2}$$

$$x \textit{ is differentiable everywhere in } J; \tag{3}$$

i.e., the derivative $\dot{x}(t)$ exists for every $t \in J$ and

$$\dot{x}(t) = f(t, x(t)) \textit{ for every } t \in J. \tag{4}$$

A function $x : J \to \mathbb{R}^n$ satisfying (2), (3) and (4) is called a *solution* of (1) and of course the properties are satisfied componentwise; i.e., if $x = (x_1, \dots, x_n)$ then (3) means that all x_k, $k = 1, \dots, n$ are differentiable and (4) reads

$$\dot{x}_k(t) = f_m(t, x_1(t), \dots, x_n(t)) \text{ for } t \in J \text{ and } k = 1, \dots, n,$$

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