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ON THE EXISTENCE OF NODAL SOLUTIONS OF THE EQUATION $-\Delta \mathbf{u} = |\mathbf{u}|^{2^*-2}\mathbf{u}$ WITH DIRICHLET BOUNDARY CONDITIONS*

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Abstract. In this paper we study the existence of nodal solutions of the equation $-\Delta u = |u|^{2^*-2}u$ with homogeneous Dirichlet boundary conditions. We first derive some compactness lemmata for Palais-Smale sequences of functions which change sign and then apply these results to obtain an existence theorem in a symmetric domain.

Introduction. In this paper we study the existence of solutions which change sign of the homogeneous Dirichlet problem

$$\begin{cases} -\Delta u = |u|^{2^*-2}u \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with smooth boundary and $2^* = \frac{2N}{N-2}$.

The solutions of (1) correspond to the critical points of the functional

$$F(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2^*}\int_\Omega |u|^{2^*}. \tag{2}$$

As is well known, the main difficulty in studying (1) or the related functional (2) is that 2^* is the critical exponent for the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^s(\Omega)$; i.e., when $s = 2^*$, such embedding is continuous but not compact. This implies that F does not satisfy the classical Palais-Smale condition and therefore the standard variational techniques do not apply directly to problem (1).

This lack of compactness makes (1) totally different from the analogous subcritical problem, since the Pohozaev identity ([17]), applied to (1) gives

$$\frac{1}{2}\int_{\partial\Omega} \left|\frac{\partial u}{\partial \nu}\right|^2 (x \cdot \nu)\, d\sigma = 0, \tag{3}$$

where ν is the outer normal to $\partial\Omega$ and (3) implies the nonexistence of positive or nodal solutions of (1) whenever Ω is strictly star-shaped.

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