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ASYMPTOTIC STABILITY OF THE EQUILIBRIUM OF THE DAMPED OSCILLATOR*

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Abstract. Conditions are given guaranteeing the property $x(t) \to 0$, $\dot{x}(t) \to 0$ $(t \to \infty)$ for every solution of the equation

$$\ddot{x} + h(t)\dot{x} + k^2 x = 0 \quad (t \geq 0,\ 0 < k = \text{const.}),$$

where h is a nonnegative function. It is known that this property requires that in the average the damping coefficient h is not "too small" or "too large". In the first part we give a necessary and sufficient growth condition on h, provided that h is not "too small" in some integral sense. Then, considering the case of small h, we show that not only the size, but the distribution of the damping "bumps" is important. The main theorem takes into account both of them. Finally, we formulate theorems for the general case when h can be both small and large. It is pointed out that the conditions restricting h above and below are interdependent.

1. Introduction.

The differential equation

$$\ddot{x} + h(t)\dot{x} + k^2 x = 0, \quad x \in \mathbb{R}, t \in \mathbb{R}_+ := [0, \infty), \tag{1.1}$$

where the function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is measurable and locally integrable, and $0 < k$ is a constant, describes the motion of an oscillator under the action of viscous friction with time-varying damping coefficient $h(t)$. Numerous papers [1–3, 6–12] (see also the references in [4]) have been devoted to conditions of the asymptotic stability of the equilibrium state $x = \dot{x} = 0$, which for the linear equation (1.1) means that $x(t) \to 0$, $\dot{x}(t) \to 0$ as $t \to \infty$ for every solution x of (1.1). It is well-known that $0 < \underline{h} \leq h(t) \leq \bar{h} < \infty$ $(t \in \mathbb{R}_+)$ is sufficient for the asymptotic stability [8].

Considering the case when the damping coefficient is bounded from below by a positive constant, Z. Artstein and E.F. Infante proved

Theorem A. ([1]) *If $h(t) \geq \underline{h} > 0$ and*

$$\int_0^t h(s)\,ds \leq Bt^2 \quad \textit{for all } t \in [0, \infty), \tag{1.2}$$

with some constants $\underline{h}$ and B, then the zero solution of (1.1) is asymptotically stable.

A necessary and sufficient condition with a more complicated integral condition is also known:

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