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THE EXISTENCE OF NONOSCILLATORY SOLUTIONS FOR FIRST AND SECOND ORDER DELAY DIFFERENTIAL EQUATIONS

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Abstract. For the delay differential equation

$$x'(t) + a(t)x(t-r) = 0, \ \ a(t) > 0 \quad \text{for} \quad r < t < \infty,$$

if the characteristic equation $\lambda + a(t)e^{-\lambda r} = 0$ has two real roots satisfying $\alpha_i < \lambda_i(t) < \beta_i$, $i = 1, 2$ $r < t < \infty$, α_i, β_i constants, then (1) has two solutions of the form $x_i(t) = \exp \int_0^t \mu_i(s)\, ds$ with $\alpha_i < \mu_i(t) < \beta_i$ for $0 < t < \infty$. This result is then used to establish a nonoscillation-comparison method.

Similar results are obtained for the equation $x''(t) - a(t)x(t-r) = 0$.

1. Introduction. Let $x(t)$ or $x(t, \phi)$ denote a solution of the initial value problem

$$\begin{aligned} &x'(t) + a(t)x(t-r) = 0, \quad r < t < \infty \\ &x(t) = \phi(t), \quad 0 \le t \le r. \end{aligned} \tag{1}$$

Here $r > 0$ is a constant, $\phi(t)$ is continuous for $0 \le t \le r$, and $a(t)$ is positive and continuous for $r \le t < \infty$.

If $a(t) = a$ is constant we shall call the equation (1c). Then (1c) will have a solution of the form $x(t) = \exp(\lambda t)$ if and only if

$$C(\lambda) \equiv \lambda + a \exp(-\lambda r) = 0. \tag{2}$$

The graph of $C(\lambda)$ looks roughly like a parabola with a minimum at $(r^{-1}\log(ar), r^{-1}\log(aer))$. If $aer < 1$, (2) has two real roots; if $aer = 1$, (2) has one real root; and if $aer > 1$, (2) has no real roots. In each case there is also a sequence of roots $\lambda_n = \mu_n + i\nu_n$, $n \ge 3$, and its complex conjugate, with $\mu_3 > \cdots > \mu_n \cdots \to -\infty$ as $n \to \infty$.

If $aer < 1$ and we call the real roots λ_1 and λ_2 then

$$0 > \lambda_1 > -r^{-1} > r^{-1} \log ar > \lambda_2 > \mu_3. \tag{3}$$

(Wright [5] discusses the distribution of these roots.)

For equation (1c) this means that there are two, one, or zero linearly independent nonoscillatory solutions and infinitely many oscillatory solutions. In this paper we show that the nonoscillatory solution property carries over to variable $a(t)$. Specifically,

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