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SADDLE-POINTS AND EXISTENCE-UNIQUENESS FOR EVOLUTION EQUATIONS

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Abstract. Certain variational and saddle-point characterizations of solutions of nonlinear evolution equations are described and analyzed. In particular existence-uniqueness theorems for the evolution equations are obtained as a consequence of the existence of a unique saddle point of an appropriate convex-concave functional.

1. Introduction. Some years ago Brezis and Ekeland [3] developed variational principles for certain classes of nonlinear evolution equations. They constructed a non-local, nonnegative functional for certain classes of initial value problems which was zero precisely at the solutions of the problem. They commented on their inability to use their variational principle to obtain direct existence results. Many aspects of their analysis were extended in [1], but the work there also did not provide existence theorems.

Here we shall extend the Brezis-Ekeland variational principle and show how solutions of these initial value problems may be characterized as saddle points of a convex-concave function $\mathcal{L}$. This functional does not require knowledge of the Green's function as in [3] and, for second-order nonlinear parabolic equations, is an ordinary integral of the unknown functions and their first derivatives. Under some natural structural assumptions, direct existence theorems for these saddle points will be obtained. Then we use the saddle point equations to show that these saddle points must have the form $(\hat{u}, \hat{u})$ where $\hat{u}$ is the unique solution of the original initial value problem.

The results described here parallel the existence-uniqueness for autonomous equations obtained using the theory of nonlinear semigroups as described in [2] and many other places. Galerkin methods as described in [6], Chapter 30 can also be used to obtain many of these results under similarly weak regularity assumptions on the coefficients of the operators involved.

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2. Variational formulations of initial value problems. We shall first describe the variational characterization due to Brezis and Ekeland [3] of the solutions

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