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ON THE ZEROS OF ASSOCIATED POLYNOMIALS OF CLASSICAL ORTHOGONAL POLYNOMIALS*

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Abstract. We establish some general properties of the associated polynomials $r_{n-1}(x)$ of classical orthogonal polynomials $p_n(x)$. As a consequence of our results we prove a conjecture recently formulated by A. Ronveaux on the location of the zeros of $r_{n-1}(x)$ and $p'_n(x)$.

1. Introduction. For $n = 0, 1 \ldots$ we denote by $p_n(x)$ any classical orthogonal polynomial (Jacobi, Hermite, and Laguerre) of degree n, on the interval (a, b) with the weight $\mu(x)$ where

$$\mu(x) = \begin{cases} (1-x)^\alpha(1+x)^\beta, \ -1 < x < 1, \ \alpha > -1, \ \beta > -1 & \text{in Jacobi case,} \\ x^\alpha e^{-x}, \ 0 < x < \infty, \ \alpha > -1 & \text{in Laguerre case,} \\ e^{-x^2}, \ -\infty < x < \infty & \text{in Hermite case.} \end{cases}$$

It is well known that the classical orthogonal polynomials satisfy the second-order differential equation

$$\sigma(x)y'' + \tau(x)y' + \lambda_n y = 0, \tag{1.1}$$

where $\sigma(x)$ is a polynomial in x of the second degree at most, $\tau(x)$ is a polynomial of the first degree, and λ_n is a constant depending on n.

We define the associated polynomial $r_{n-1}(x)$ of $p_n(x)$ by means of the integral

$$r_{n-1}(x) = \frac{1}{c_0}\int_a^b \frac{p_n(x) - p_n(t)}{x-t}\mu(t)\,dt, \quad n = 1, 2, \ldots, \tag{1.2}$$

where

$$c_0 = \int_a^b \mu(t)\,dt.$$

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