Differential and Integral Equations, Volume 6, Number 5, September 1993, pp. 1125–1135.

APPLICATIONS OF MORSE THEORY TO BIFURCATION THEORY

SŁAWOMIR RYBICKI
Technical University of Gdańsk, Department of Mathematics
80–952 Gdańsk, ul. Majakowskiego 11/12, Poland

(Submitted by: Jean Mawhin)

Abstract. The topological structure of the set of bifurcation points of the boundary value problem (BVP) of the form

$$\begin{cases} \dot{x}(t) = g(t, x(t), \lambda) \\ x(0) = x(1) \end{cases}$$

is studied. Sufficient conditions for the existence of bifurcation points of (BVP) are given in terms of the Brouwer topological degrees of suitable maps. Approach to bifurcation theory is proposed via Morse theory. In particular we describe the structure of the set of bifurcation points on "small spheres" centered at the origin.

1. Introduction. Consider a continuous map $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ such that $f(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}^k$. We are interested in sufficient conditions for the existence of bifurcation points for the equation $f(x, \lambda) = 0$.

Many authors have considered such problems. In the case $k = 1$ Krasnosielski (see [4]) has proved a theorem which gives, in terms of the Brouwer topological degree, sufficient conditions for the existence of a bifurcation point of $f(x, \lambda) = 0$. This theorem is a very useful tool in bifurcation theory. One can find generalizations of the classical Krasnosielski Theorem, for example, in [7], [10], [11].

In [1] Alexander has constructed an invariant whose nontriviality implies the existence of a bifurcation point for the equation of the form $f(x, \lambda) = 0$. This invariant is an element of the group $\pi_{k-1}(GL(n))$ and generally it is difficult to verify if it is a nontrivial element in $\pi_{k-1}(GL(n))$. Of course, in the case when $\pi_{k-1}(GL(n))$ is a trivial group the Alexander theorem does not work.

On the other hand multiparameter bifurcation theorems have been proved for example in [3].

The goal of this paper is to construct sufficient conditions for the existence of a bifurcation point of $f(x, \lambda) = 0$ in the case when the dimension of the parameter space is greater than one and when the Alexander invariant can not be applied.

Assume that f is a C^1-map and define a map $\Phi : \mathbb{R}^k \to \mathbb{R}$ by the formula $\Phi(\lambda) = \det(D_x f(0, \lambda))$. It follows from the Krasnosielski theorem that if the map Φ changes sign at the point $\lambda_0 \in \mathbb{R}^k$ then the point $(0, \lambda_0) \in \mathbb{R}^n \times \mathbb{R}^k$ is a bifurcation point for the equation $f(x, \lambda) = 0$.

Received January 1992.
AMS Subject Classification: 34B15, 34C25, 58F14.