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OPTIMAL GRADIENT BOUNDS AND HEAT EQUATION*

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Abstract. Let $u_{q,h}$ be the solution to the heat equation $\frac{\partial}{\partial t}u - \Delta u = q(x,t)$ in $\mathbb{R}^n \times (0,T]$ satisfying $u(x,0) = h(x)$ on $\mathbb{R}^n$. We find the maximum of $\|Du_{q,h}(\cdot,T)\|_{L^\infty(\mathbb{R}^n)}$ in terms of the decreasing rearrangements of q and h. The techniques are based on the study of level sets.

1. Introduction. The background of the present paper are a priori estimates for the gradient of solutions to linear parabolic equations. Bounds for L^p norms of the gradient of such solutions in terms of the data are classic (see e.g. [4]). On the other hand, not much seems to be known about the optimal form of this kind of estimates. In [5] sharp inequalities involving L^2 norms have been proved. As far as we know, the question for powers p greater than 2 is still open. Our purpose here is to study the problem for the heat equation when $p = \infty$. More precisely, consider the following Cauchy problem:

$$\begin{cases} \frac{\partial}{\partial t}u - \Delta u = q(x,t) & \text{in } \mathbb{R}^n \times (0,T] \\ u(x,0) = h(x) & \text{on } \mathbb{R}^n. \end{cases} \tag{1.1}$$

Here, $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial {x_i}^2}$, the Laplace operator, and the data q and h are real-valued functions on $\mathbb{R}^n \times (0,T]$ and on $\mathbb{R}^n$, respectively.

We deal with the customary solution $u_{q,h}$ to problem (1.1) given by

$$u_{q,h}(x,t) = \int_0^t \int_{\mathbb{R}^n} K(x-y,t-\tau)q(y,\tau)\,dy\,d\tau + \int_{\mathbb{R}^n} K(x-y,t)h(y)\,dy, \tag{1.2}$$

where $K(x,t) = (4\pi t)^{-n/2}\exp(-\frac{|x|^2}{4t})$, the heat kernel.

According to a classical model for the heat conduction $u_{q,h}(x,t)$ represents the temperature at point x and at time t of a homogeneous isotropic solid medium occupying the whole space, h is the initial temperature and q stands for density of power of a heat source or sink, depending on whether $q > 0$ or $q < 0$. Moreover, $Du_{q,h}$, the gradient of $u_{q,h}$ with respect to the x variables, is the negative of the heat flux.

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