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SYMMETRY RESULTS FOR REACTION-DIFFUSION EQUATIONS*

HANS G. KAPER, MAN KAM KWONG, YI LI†
Mathematics and Computer Science Division
Argonne National Laboratory, Argonne, IL 60439

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Abstract. This article is concerned with symmetry properties of the solutions of the reaction-diffusion equation $\Delta u + f(u) = 0$ in a bounded connected domain Ω in $\mathbb{R}^N$ ($N = 2, 3, \dots$). Of especial interest are nonlinear source terms f of the type $f(u) = u^p - u^q$ with $0 \le q < p \le 1$. Two results are presented. The first result concerns the solution of a free boundary problem, where the domain Ω is unknown and u and its normal derivative $\partial_n u$ are required to vanish on the boundary $\partial\Omega$ of Ω. It is shown that, if f is the sum of a continuous nondecreasing function and a Lipschitz continuous function on $[0, \infty)$, then the free boundary problem does not have a positive solution unless Ω is a ball; in this case, any positive solution is radially symmetric around the center of the ball and decreasing with the radial distance from the center. The second result concerns the solution of the Dirichlet problem on a ball in $\mathbb{R}^N$, when the nonlinear source term f is continuous, but not necessarily Lipschitz continuous at 0. It is shown that, if f is the sum of a locally Lipschitz continuous function on $(0, \infty)$ that is nonincreasing near 0 and a function that is Lipschitz continuous on $[0, \infty)$, then any positive solution u is radially symmetric around the center of the ball and decreasing with the radial distance from the center.

1. Statement of the problem. In this article we present two symmetry results for positive solutions of the reaction-diffusion equation

$$\Delta u + f(u) = 0, \quad x \in \Omega, \tag{1.1}$$

where Ω is a bounded connected domain in $\mathbb{R}^N$ ($N = 2, 3, \dots$). In the first problem, Ω is not specified, but both u and its normal derivative $\partial_n u$ are required to vanish on the boundary $\partial\Omega$ of Ω. In the second, Ω is a ball in $\mathbb{R}^N$, and u is required to vanish on the boundary of the ball.

The first problem would clearly be overdetermined if Ω were specified. However, as Ω is left unspecified, the extra degree of freedom may be enough to allow for a solution—which, in this case, consists of the pair (Ω, u). We refer to this problem as the *free boundary problem*. Of particular interest is the question of whether a solution of the free boundary problem, if it exists, has any symmetry properties.

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†Permanent address: Department of Mathematics, University of Rochester, Rochester, NY 14620. Supported in part by the National Science Foundation Grant DMS-8903192-A01.
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