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ON THE SYMMETRY OF MINIMIZING HARMONIC MAPS IN N DIMENSIONS

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Introduction. Let Ω be a domain in $\mathbb{R}^{k+n}$. We say that Ω is $\mathbf{S}^{n-1}$-symmetric (or symmetric) if it can be written, for some domain ω in $\mathbb{R}_+^{k+1}$, as

$$\Omega = \{(x, ry) : (x, r) \in \omega,\ y \in \mathbf{S}^{n-1}\},$$

where $\mathbb{R}_+^{k+1} = \{(x, r) : x \in \mathbb{R}^k,\ r \in \mathbb{R}_+\}$. A map u defined on such a domain, with values in $\mathbf{S}^n$, is said to be symmetric if there exists a function $\varphi : \omega \to \mathbb{R}$ such that

$$u(x, ry) = (y \cos\varphi(x, r), \sin\varphi(x, r)), \quad \forall (x, ry) \in \Omega.$$

A minimizing harmonic map (or a minimizer) for a boundary data $\Psi \in H^{\frac{1}{2}}(\partial\Omega, \mathbf{S}^n)$ is map $u : \Omega \to \mathbf{S}^n$ whose Dirichlet energy $E(u) = \int_\Omega |\nabla u|^2 \, dV$ is minimal among the maps $v \in H^1(\Omega, \mathbf{S}^n)$ satisfying $v = \Psi$ on $\partial\Omega$. The symmetry problem for minimizing harmonic maps can be stated as follows: is a minimizer for a symmetric boundary data $\Psi : \partial\Omega \to S^n$ necessarily symmetric? Although the answer to this question is negative in general (see Remark 1 below), it follows from results of Jäger and Kaul (see [5, 6]) that for a symmetric $\Psi : \partial\Omega \to S^n$ with image lying in a compact subset of a hemisphere of $\mathbf{S}^n$, there is a unique minimizer, which is in particular symmetric. Our main theorem states that this last result still holds, with an obvious modification of the uniqueness statement, when we only assume that $\Psi(\partial\Omega)$ is contained in a closed hemisphere of $\mathbf{S}^n$; i.e., in $\mathbf{S}^n_+ = \mathbf{S}^n \cap \{x_{n+1} \geq 0\}$ or $\mathbf{S}^n_- = \mathbf{S}^n \cap \{x_{n+1} \leq 0\}$.

We remark that this result was known for the case $n = 2$, $k = 0$. Indeed, the case where Ω is a disk follows from a result of Brezis and Coron [2], see also [1], while the case where Ω is an annulus was proved by Sandier [9, 10]. Another proof was then given by Kaniel and Shafrir in [7], and our proof of Theorem 1 below uses a generalization of their method. Combining our result with that of [6] we conclude that for $n \geq 7$ the map $(\frac{x}{|x|}, 0)$ from $\mathbf{B}^n$ to $\mathbf{S}^n$ is the *unique* minimizer for its boundary values. However, our method does not apply to get the result (due to Brezis-Coron-Lieb [3] for the case $n = 3$ and to F.H. Lin [8] for the general case $n \geq 3$), that the map $\frac{x}{|x|}$ from $\mathbf{B}^n$ to $\mathbf{S}^{n-1}$ is the unique minimizer for its boundary values (we get it only for $n \geq 7$).

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