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WEAKLY NONLINEAR LARGE TIME BEHAVIOR IN SCALAR CONVECTION-DIFFUSION EQUATIONS

ENRIQUE ZUAZUA†
Departamento de Matemática Aplicada, Universidad Complutense, 28040 Madrid, Spain

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Abstract. The large time behavior of scalar convection-diffusion equations

$$u_t - \Delta u = \vec{a} \cdot \nabla(|u|^{q-1}u) \quad \text{in } \mathbb{R}^N \times (0, \infty)$$

with initial data in $L^1(\mathbb{R}^N)$ is studied. The case where $q > 1 + 1/N$ is considered. It is by now well known that, in this range of exponents q, solutions behave as the heat kernel as $t \to \infty$. We make more precise this "weakly nonlinear" large time behavior obtaining the second term in the asymptotic development of solutions. The following three cases are distinguished: (a) $q \in (1 + \frac{1}{N}, 1 + \frac{2}{N})$, (b) $q = 1 + \frac{2}{N}$ and (c) $q > 1 + \frac{2}{N}$. The second term in the asymptotic development is of different nature in each of these three cases. In particular, the momentum $\int_{\mathbb{R}^N} x\varphi(x)\, dx$ of the initial data only appears in this second term if $q > 1 + \frac{2}{N}$. The proofs combine scaling arguments with the use of the similarity variables associated to the heat equation.

1. Introduction. In this paper we study the large time behavior of solutions of the scalar equation

$$u_t - \Delta u = \vec{a} \cdot \nabla(|u|^{q-1}) \quad \text{in } \mathbb{R}^N \times (0, \infty) \tag{1}$$

$$u(x, 0) = \varphi(x) \in L^1(\mathbb{R}^N), \tag{2}$$

where $\vec{a} \in \mathbb{R}^N$, $q > 1 + \frac{1}{N}$ and $\cdot$ denotes the scalar product in $\mathbb{R}^N$. This equation represents a very simple model of diffusion and convection.

For every $\varphi \in L^1(\mathbb{R}^N)$, system (1)–(2) admits a unique solution $u \in C([0, \infty); L^1(\mathbb{R}^N))$. This solution is smooth for $t > 0$. Integrating equation (1) in $\mathbb{R}^N$ we deduce that the mass of solutions is conserved, i.e.,

$$\int_{\mathbb{R}^N} u(x,t)\, dx = \int_{\mathbb{R}^N} \varphi(x)\, dx, \quad \forall t > 0. \tag{3}$$

On the other hand, the following L^p-estimates hold (cf. [5]):

$$\begin{cases} \text{(i)} \quad \|u(t)\|_p \le C_p \|\varphi\|_1 t^{-\frac{N}{2}(1-\frac{1}{p})}, \quad \forall t \ge 0 \\ \text{(ii)} \quad \|\nabla u(t)\|_p \le C_p \|\varphi\|_1 t^{-\frac{N}{2}(1-\frac{1}{p})-\frac{1}{2}}, \quad \forall t \ge 1 \end{cases} \tag{4}$$

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