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A UNIQUENESS RESULT FOR MEASURE-VALUED SOLUTIONS OF NONLINEAR HYPERBOLIC EQUATIONS

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Abstract. We will study the solutions of the nonlinear hyperbolic equation $u_t + \mathrm{div}(\mathbf{v}f(u)) = 0$ in $\mathbb{R}^N \times [0,T]$, with given initial condition $u(\cdot,0) = u_0(\cdot)$ in $\mathbb{R}^N$, where $\mathbf{v}$ is a given function from $\mathbb{R}^N \times [0,T]$ to $\mathbb{R}^N$ and f is a given function from $\mathbb{R}$ to $\mathbb{R}$. We will prove that if ν is a measure-valued solution satisfying some entropy condition, which we define, then $\nu_{x,t} = \delta_{u(x,t)}$ for almost every $(x,t) \in \mathbb{R}^N \times [0,T]$, where u is the unique entropy weak solution to the equation.

We consider the following nonlinear hyperbolic equation, with the initial condition

$$\begin{cases} u_t(x,t) + \mathrm{div}(\mathbf{v}f(u(x,t))) = 0, & x \in \mathbb{R}^N, \ t \in [0,T], \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1}$$

where $T > 0$, u_t denotes the derivative of u with respect to t, $\mathrm{div}\,\mathbf{v} = \sum_{i=1}^N \partial_{x_i} v_i$, where $x = (x_1, \dots, x_N)$, $\mathbf{v} = (v_1, \dots, v_N)$, $\mathbf{v} \in C^3(\mathbb{R}^N \times [0,T], \mathbb{R}^N)$. It is assumed that there exists $V > 0$ such that $\sup\{|\mathbf{v}(x,t)|,\ x \in \mathbb{R}^N,\ t \in [0,T]\} \le V$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$, f is a given function of class C^3 from $\mathbb{R}$ to $\mathbb{R}$, and $u_0 \in L^\infty(\mathbb{R}^N)$ is a given function. It is also assumed that $\sup\{-\mathrm{div}(\mathbf{v}(x,t))f'(u),\ x \in \mathbb{R}^N,\ t \in [0,T],\ u \in \mathbb{R}\} < \infty$. This is true, for instance, if $\mathrm{div}\,\mathbf{v} = 0$. Under these assumptions, Kruzkov [4] gives results of existence (and uniqueness) of the entropy weak solution u; i.e., a function $u \in L^\infty(\mathbb{R}^N \times]0,T[)$ which satisfies:

$$\begin{aligned} &\int_{\mathbb{R}^N} \int_0^T \eta(u(x,t))\varphi_t(x,t) + \Phi(u(x,t))\mathbf{v}(x,t) \cdot \mathrm{grad}\,\varphi(x,t))\,dt\,dx \\ &+ \int_{\mathbb{R}^N} \int_0^T \mathrm{div}\,\mathbf{v}(x,t)\varphi(x,t)\Big(\Phi(u(x,t)) - \eta'(u(x,t)f(u(x,t))\Big)dt\,dx \\ &+ \int_{\mathbb{R}^N} \eta(u_0(x))\varphi(x,0)\,dx \ge 0 \quad \forall \varphi \in C_c^1(\mathbb{R}^N \times [0,T[, \mathbb{R}_+), \end{aligned} \tag{2}$$

for any C^1 convex function η from $\mathbb{R}$ to $\mathbb{R}$, and Φ such that $\Phi' = f'\eta'$,

where $C_c^1(E,F)$ is the set of C^1 functions from E to F with compact support in E.

Other results of existence and uniqueness of a weak entropy solution to problem (1) are given in [8,] [6], and [5].

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