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SINGULAR DEGENERATE PARABOLIC EQUATIONS WITH APPLICATIONS TO GEOMETRIC EVOLUTIONS

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Abstract. We prove a comparison theorem for viscosity solutions of degenerate parabolic equations which are singular at finite directions of derivatives. We apply our theorem to construct a global generalized evolution for interface equations with a certain class of the interface energy, not necessarily C^2.

1. Introduction. We are concerned with a degenerate parabolic equation of form

$$u_t + F(\nabla u, \nabla^2 u) = 0 \quad \text{in} \quad Q = (0,T) \times \Omega, \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and $T > 0$. The function $F(p, X)$ is allowed to have singularities when p belongs to finitely many half lines ℓ_i of the form

$$\ell_i = \{\eta q_i \ ; \ \eta \geq 0\}, \quad q_i \in \mathbb{R}^n \backslash \{0\}, \quad i = 1, \cdots, m.$$

As explained later, such an F naturally arises in a level set approach to motion of phase boundaries. Here $u_t = \partial u/\partial t, \nabla u$ and $\nabla^2 u$ denote, respectively, the time derivative of u, the gradient of u and the Hessian of u in space variables.

Our first goal is to establish a comparison principle for viscosity solutions of (1.1). If F has singularities only for $p = 0$, a comparison principle is established in [5], assuming that F can be extended continuously at $(p, X) = (0, O)$. See [12] for simplification of the proof. (The paper [6] includes corrections of technical errors in [5], [12]).

Although we still appeal to Crandall-Ishii's lemma [7], the method in [12] or [5] does not apply to our setting, because F has singularities other than $p = 0$. By a clever choice of "test function", we shall prove a comparison principle under assumptions on the value of semicontinuous envelope of F at $(\mu q_i, \nu q_i \otimes q_i)$, $\mu > 0$, $\nu \in \mathbb{R}$, where $\otimes$ denotes the tensor product.

Our second goal is to apply our comparison results to geometric evolutions. Let Γ_t denote the hypersurface expressed as the boundary of a bounded open set D_t in $\mathbb{R}^n$ ($n \geqq 2$) at time t. Let $\mathbf{n}$ denote the unit exterior normal vector field on $\Gamma_t = \partial D_t$. Let $V = V(t, x)$ denote the speed of Γ_t at $x \in \Gamma_t$ in the exterior normal direction. The geometric evolution of Γ_t studied in [2], [3] is of the form

$$V = \frac{1}{\beta(\mathbf{n})}\Big(-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big(\frac{\partial H}{\partial p_i}(\mathbf{n})\big) + c\Big), \tag{1.2}$$

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