Differential and Integral Equations, Volume 6, Number 6, November 1993, pp. 1253–1264.

REMARKS ON THE ASYMPTOTIC BEHAVIOUR FOR ELLIPTIC EQUATIONS WITH CRITICAL GROWTH

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(Submitted by: Roger Nussbaum)

Abstract. We discuss the asymptotic behaviour of positive solutions u_ϵ to the Dirichlet problem: $-\Delta u - \lambda(\epsilon)u = N(N-2)u^{p-\epsilon}$ in the unit ball, where $p = (N+2)/(N-2)$ is the critical Sobolev exponent. We show that the rate of convergence of u_ϵ as $\epsilon \to 0$ varies widely with suitable $\lambda(\epsilon)$.

1. Consider the problem

$$\text{(I)} \quad \begin{cases} -\Delta u - \lambda(\varepsilon)u = N(N-2)u^{p-\varepsilon} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is the unit ball in $\mathbf{R}^N$ with $N \geqq 3$, $p = (N+2)/(N-2)$, $\varepsilon \geqq 0$ and $\lambda(\varepsilon)$ is a given function of ε.

We recall some known results:

(1) Every solution of (I) is radially symmetric and decreasing (see [5]).
(2) If $\varepsilon > 0$, problem (I) has a solution u_ε for any $\lambda(\varepsilon) < \lambda_0$. If $\varepsilon = 0$ and if $N \geqq 4$, problem (I) has a solution if and only if $\lambda(0) \in (0, \lambda_0)$, but if $N = 3$, it has one if and only if $\lambda(0) \in (\frac{1}{4}\lambda_0, \lambda_0)$. Here, λ_0 denotes the principal eigenvalue of $-\Delta$ on the unit ball (see [3]).

Here we want to discuss the asymptotic behaviour of u_ε as $\varepsilon \to 0$. In [2], Atkinson and Peletier, using ODE methods, made the first study when $\lambda(\varepsilon) \equiv 0$. There they showed that any solution u_ε of (I) has the following limiting behaviour:

$$\lim_{\varepsilon\to 0} \varepsilon^{1/2}\cdot u_\varepsilon(0) = \Big(\frac{2\sigma_N}{N-2}\Big)^{1/2}\Big(\frac{N(N-2)}{S_N}\Big)^{N/4}$$
$$\lim_{\varepsilon\to 0} \varepsilon^{-1/2}\cdot u_\varepsilon(x) = \Big(\frac{2\sigma_N}{N-2}\Big)^{-1/2}\Big(\frac{N(N-2)}{S_N}\Big)^{-N/4}(N-2)\sigma_N G(x) \quad \text{at any } x \neq 0,$$

where we denote by $G(x)$ the Green's function of $-\Delta$ on the unit ball, σ_N is the area of the unit sphere in $\mathbf{R}^N$ and S_N is the best Sobolev constant:

$$\sigma_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}, \qquad S_N = \pi N(N-2)\left(\frac{\Gamma(N/2)}{\Gamma(N)}\right)^{2/N}.$$

Received for publication August 1992.
AMS Subject Classifications: 35B05.