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ON HARNACK'S INEQUALITY FOR A CLASS OF STRONGLY DEGENERATE SCHRÖDINGER OPERATORS FORMED BY VECTOR FIELDS

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1. Introduction. Let $X_1, \dots, X_m$ be real C^∞ vector fields on $\mathbb{R}^d$ ($d \geq 3$) satisfying Hörmander's condition of type s, i.e., $X_1, , \dots, X_m$ and their commutators up to order s span the tangent space of $\mathbb{R}^d$ at each point of $\mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ be an open and connected domain. As studied in [15], we can define a metric ϱ on Ω associated to the vector fields. Moreover, the doubling property holds on (Ω, ϱ); i.e.,

$$|B(x, 2\delta)| \leq C|B(x, \delta)|,$$

for any $x \in E \Subset \Omega$ and $\delta > 0$. Thus (Ω, ϱ) is a homogeneous metric space in the sense of [6].

We say a locally integrable nonnegative function $w(x)$ is in the class of $A_2(\mathbb{R}^d, \varrho)$, or $w \in A_2$, if

$$\sup_{B \subset \mathbb{R}^d} \frac{1}{|B|} \int_B w(x)\, dx \cdot \frac{1}{|B|} \int_B w(x)^{-1} dx \leq c_w$$

with c_w independent of the metric balls $B \subset \mathbb{R}^d$. This c_w is called the A_2 constant of w. If the above inequality only holds for all balls $B \subset \Omega$, then we say $w \in A_2(\Omega)$.

We now state the weighted Poincaré and Sobolev inequalities for vector fields satisfying Hőrmander's condition proved in [13]. Let $w \in A_2(\Omega)$, $E \Subset \Omega$. Then there exist constants $C > 0$, $r_0 > 0$ and $\tau > 2$ such that for any metric ball $B = B(x, r) \subset \Omega$, $x \in E$, $r \leq r_0$ and $f \in C^\infty(\overline{B})$, we have

$$\Big(\frac{1}{w(B)} \int_B |f - f_B|^\tau w\Big)^{1/\tau} \leq Cr\Big(\frac{1}{w(B)} \int_B \sum_{i=1}^m |X_i f|^2 w\Big)^{1/2}. \tag{1.1}$$

For any $f \in C_0^\infty(B)$, we have

$$\Big(\frac{1}{w(B)} \int_B |f|^\tau w\Big)^{1/\tau} \leq Cr\Big(\frac{1}{w(B)} \int_B \sum_{i=1}^m |X_i f|^2 w\Big)^{1/2}, \tag{1.2}$$

where $w(B) = \int_B w(x) dx$.

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