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JENSEN'S INEQUALITY IN THE CALCULUS OF VARIATIONS

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Abstract. A general, unified approach to lower semicontinuity with respect to weak and biting convergence is introduced. It is based on Jensen's inequality for parameterized measures. We recover in this frame most of the now classical weak lower semicontinuity results.

1. Introduction. Jensen's inequality has lately received some attention for it appears to be closely connected not only to the usual notion of convexity but also to much more general kinds of convexity. Since these are the basic constitutive assumptions for lower semicontinuity results, we somehow expect to relate both. This is the goal of this paper. As a matter of fact, this standpoint opens the gate to a different way of understanding weak lower semicontinuity based on Jensen's inequality, which might be useful in more general situations than the ones described here (see [14]). The principal ingredient in all this is the concept of parameterized measure or Young measure. When we talk about Jensen's inequality, we mean Jensen's inequality with respect to this parameterized measure. These were originally introduced as a tool to deal with non-convex problems in the Calculus of Variations. We show that they are very helpful in working with regular variational principles as well. Indeed, the structure of Young measures is connected to weak lower semicontinuity. Some basic references are [3, 5, 7, 20, 23]. Previous work which greatly motivated this point of view is Kinderlehrer, Pedregal [15–17].

Assume we have a sequence of measurable functions, $\{z_j\}$, bounded in some space $L^p(\Omega)$ for some regular open bounded set Ω. Then the sequence of pth powers, $\{|z_j|^p\}$, is bounded in $L^1(\Omega)$. Under these circumstances, there is a subsequence of the z_j's, not relabeled, and a family of probability measures, $\{\nu_x\}_{x\in\Omega}$, the corresponding parameterized measure, such that whenever the composites $\varphi(z_j)$ converge weakly in $L^1(\Omega)$ they do towards the function

$$\overline{\varphi}(x) = \int_{\mathbf{R}^m} \varphi(\lambda)\, d\nu_x(\lambda).$$

This means

$$\lim_{j\to\infty} \int_E \varphi(z_j)\, dx = \int_E \overline{\varphi}\, dx, \tag{1.1}$$

for any measurable $E \subset \Omega$. We assume that the functions z_j take values in $\mathbf{R}^m$. Usually the main difficulty to be overcome is to make sure that for a particular

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