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ON THE UNIQUENESS AND NONUNIQUENESS OF WEAK SOLUTIONS OF HYPERBOLIC-PARABOLIC VOLTERRA EQUATIONS

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Abstract. The uniqueness and nonuniqueness of weak solutions of the equation $u_t(t,x) = \int_0^t k(t-s)(\sigma(u_x))_x(s,x)\,ds + f(t,x)$ is studied. Weak solutions are unique if $\Re\hat{k}(z) \ge \kappa|\Im\hat{k}(z)|$ and σ is increasing with $\sup\{\sigma'(p)\}/\inf\{\sigma'(p)\} < 1 + 2\kappa(\sqrt{\kappa^2+1}+\kappa)$. If $k(0)=1$, $k' \in BV_{\mathrm{loc}}(\mathbb{R}^+)$, and $\sigma \in C^2(\mathbb{R})$ with σ' strictly positive and not a constant, then there can be infinitely many weak solutions.

1. Introduction. The purpose of this paper is to study under what assumptions on the kernel k and the nonlinearity σ one can show that a weak solution of the equation

$$\begin{aligned} u_t(t,x) &= \int_0^t k(t-s)\big(\sigma(u_x)\big)_x(s,x)\,ds + f(t,x), \quad t \ge 0, \quad x \in (0,1), \\ u(0,x) &= u_0(x), \quad x \in [0,1], \\ u(t,0) &= u(t,1) = 0, \quad t > 0, \end{aligned} \tag{1}$$

(where the subscripts denote differentiation with respect to the corresponding variables), is unique and when one can show that there are several different weak solutions.

The reasons for considering equation (1) are that it appears in mathematical models of viscoelasticity, see [6] and [13], and that it is (in some sense) an interpolation of a parabolic and a hyperbolic equation. This is due to the fact that when $k \equiv 1$ we get the nonlinear wave equation

$$u_{tt}(t,x) = \big(\sigma(u_x)\big)_x(t,x) + f_t(t,x),$$

and if, on the other hand, k is replaced by the delta functional, then we get the nonlinear diffusion equation

$$u_t(t,x) = \big(\sigma(u_x)\big)_x(t,x) + f(t,x).$$

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