Differential and Integral Equations, Volume 7, Number 2, March 1994, pp. 345–366.

STABILITY OF WAVE EQUATIONS WITH DISSIPATIVE BOUNDARY CONDITIONS IN A BOUNDED DOMAIN

ANDREAS WYLER

Basler, Versicherungs-Gesellschaft, Aeschengraben 21, 4002 Basel, Switzerland

(Submitted by: Klaus Schmitt)

Abstract. In this paper we shall treat stability questions for the wave equation with variable coefficients on a bounded domain. The boundary $\partial\Omega$ is subdivided into an energy reflecting part Γ and a nonempty absorbing part Σ. We give criteria for asymptotic and exponential energy decay. In special cases similar criteria have already been given by other authors. Our methods make new use of some fairly recent theorems in semigroup theory.

1. Introduction: Formulation of the problem and statement of main results. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain of class C^2. Consider a disjoint partition of $\partial\Omega$ into two subsets Γ and Σ, each of them open and closed in $\partial\Omega$ (cf. Remark 1.2 for more general partitions). Assume throughout that $\Sigma \neq \emptyset$. The IBVP (Initial Boundary Value Problem) we aim to investigate in this paper is the so called *Wave equation with dissipative boundary conditions*, namely the problem

$$\rho(x)\ddot{u}(t,x) + \mathcal{A}u(t,x) = 0; \quad (t,x) \in [0,\infty) \times \Omega \tag{1.1}$$

$$u(t,s) = 0; \quad (t,s) \in [0,\infty) \times \Gamma \tag{1.2}$$

$$\sigma(s)\dot{u}(t,s) + \mathcal{B}u(t,s) = 0; \quad (t,s) \in [0,\infty) \times \Sigma \tag{1.3}$$

$$u(0,x) = u_0(x)\ ,\ \dot{u}(0,x) = v_0(x); \quad x \in \Omega. \tag{1.4}$$

As a concrete example we could consider the case $\rho = 1$ in Ω, $\sigma = \text{const.} > 0$ on Σ, $\mathcal{A} = -\Delta$ and $\mathcal{B} = \partial/\partial\nu$, where $\partial/\partial\nu$ denotes the derivative along the outer normal vector field ν on $\partial\Omega$. The dissipative boundary condition (1.3) arises naturally in some optical and acoustical problems, cf. [2]. In order to get an idea of what happens to a wave after reflection on the boundary part Σ the reader may consider solutions of the form $u(t,x) = f(x-t) + g(x+t)$ in the case of one space variable and for $(\mathcal{A},\mathcal{B})$ as in the previous example.

More generally, we consider coefficients and differential operators as follows. Let

$a_{ij} \in C^1(\bar{\Omega},\mathbb{R});\ i,j = 1,\dots,n$ be such that

$a_{ij} = a_{ji};\ i,j = 1,\dots,n$ and $a_{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2;\ x \in \Omega,\ \xi = (\xi_i) \in \mathbb{R}^n$

for some constant $a_0 > 0$

(we use the summation convention throughout). For abbreviation, we write $a(x)$ for the symmetric $n \times n$-matrices $(a_{ij}(x))_{1\leq i,j\leq n}$ and a for the function $[x \mapsto a(x)]$.

Received October 1992.

AMS Subject Classifications: 35L05, 35B40, 47D05.