Differential and Integral Equations, Volume 7, Number 2, March 1994, pp. 301–313.

NONEXISTENCE OF A POSITIVE SOLUTION OF THE LAPLACE EQUATION WITH A NONLINEAR BOUNDARY CONDITION[1]

BEI HU
Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556

(Submitted by: David Kinderlehrer)

Abstract. In this paper we prove that the only nonnegative solution to the Laplace equation $\Delta u = 0$ in the half space $\{(x_1, \cdots, x_n);\ x_1 > 0\}$ subject to the boundary condition $\frac{\partial u}{\partial n} = u^p$ on $\{x_1 = 0\}$ is the trivial solution $u \equiv 0$ when p is subcritical, namely, $1 \le p < \frac{n}{n-2}$. This result is then used to obtain the blowup rate of a heat equation with the boundary condition $\frac{\partial u}{\partial n} = u^p$.

1. Introduction. Let us first consider the following heat equation with a nonlinear boundary condition:

$$\begin{aligned} &u_t = \Delta u && \text{for } x \in \Omega,\ \ t > 0, \\ &\frac{\partial u}{\partial n} = u^p && \text{for } x \in \partial\Omega,\ \ t > 0, \\ &u(x,0) = u_0(x) && \text{for } x \in \Omega \quad (u_0(x) \ge 0). \end{aligned} \tag{1.1}$$

Throughout this paper, n denotes the exterior normal direction. It is known ([13], [14], [15]) that the solution for this problem will blow up in finite time, if $u_0(x) \not\equiv 0$. In the one space dimensional case as well as a radial symmetric domain in $\mathbb{R}^n$, the blowup set and the blowup rate were obtained ([9], [3]) under certain assumptions on the initial data.

For several space dimensions, the problem is much more challenging. Using the integral equation method, partial results were obtained in [16]. In our recent paper [11], the blowup rate is established, under the monotonicity assumption $\Delta u_0(x) \ge 0$ and the restriction $1 < p < \frac{n-1}{n-2}$; some asymptotic behavior is also established. Let us also mention some other related work [4], [8] and [12].

The proof in [11] uses the nonexistence of a nontrivial nonnegative solution to the following elliptic problem:

$$\Delta u = 0 \quad \text{in } \{(x_1, \dots, x_n);\ x_1 > 0\}, \tag{1.2}$$

$$\frac{\partial u}{\partial n} = u^p \quad \text{on } \{x_1 = 0\}. \tag{1.3}$$

Received for publication September 1993.

[1]The author is partially supported by National Science Foundation Grant DMS 92-24935. This work is carried out while the author visits the Northwestern University; he would like to thank Professor Emmanuele DiBenedetto and the Mathematics Department for their hospitality.

AMS Subject Classifications: 35J05, 35J65, 35K05, 35K60.