Differential and Integral Equations, Volume 7, Number 6, November 1994, pp. 1649–1707.

FINSLER STRUCTURES FOR THE PART METRIC AND HILBERT'S PROJECTIVE METRIC AND APPLICATIONS TO ORDINARY DIFFERENTIAL EQUATIONS

ROGER D. NUSSBAUM*
Mathematics Department, Rutgers University, New Brunswick, NJ 08903

Dedicated to the memory of Peter Hess

0. Introduction. Over the past thirty years, a powerful theory of monotone dynamical systems has been developed by many authors. A *partial* list of contributors would include N. Alikakos, E. N. Dancer, M. Hirsch, P. Hess, M.A. Krasnoselskii, U. Krause, H. Matano, P. Polacik, H.L. Smith, P. Takăč and H. Thieme. If one understands the subject more generally as a chapter in the study of linear and nonlinear operators which map a subset of a "cone" C_1, into a cone C_2, then the relevant literature, encompassing as it does the beautiful classical theory of positive linear operators, is enormous. Usually, in the study of monotone dynamical systems, it has been assumed that the map or flows in question are "strongly monotone." In this paper we shall try to show that a significant part of this theory does not depend on monotonicity, and is a special case of results about maps T which take a metric space (M, ρ) into itself and satisfy

$$\rho(T(x), T(y)) < \rho(x, y) \quad \text{for all } x \neq y \quad \text{or} \tag{0.1}$$

$$\rho(T(x), T(y)) \leq \rho(x, y) \quad \text{for all } x, y, \tag{0.2}$$

or sometimes are just assumed Lipschitzian. We will usually assume that M is a subset of a cone C in a vector space and ρ will be either the "part metric" or "Hilbert's projective metric" (see Section 1 below). We specifically emphasize that we allow equation (0.2) (so T is "nonexpansive") and that in this case there are many intriguing open and apparently difficult questions concerning the behaviour of iterates of T: see Section 3 below.

As we have already remarked, the assumption of strong monotonicity has usually been made; and in many applications this is a natural assumption. It seems less widely known that there are important applications where strong monotonicity fails and where, in addition, equation (0.2), but not equation (0.1), is satisfied. To illustrate this point, we mention a class of examples which arises in statistical mechanics [14, 15, 21, 22], in machine scheduling problems [4, 16, 17] and elsewhere. Let S denote a compact Hausdorff space, $C(S) = X$, the Banach space of continuous, real-valued functions on S (in the sup norm), K the cone of nonnegative functions in $C(S)$ and $\overset{\circ}{K}$ the interior of K. If S is the set of positive integers i, $1 \leq i \leq n$, $C(S) = \mathbb{R}^n$ and

$$K := K^n := \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for } 1 \leq i \leq n\}.$$

Received for publication March 1994.
*Partially supported by NSF DMS 91-05930.