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EXISTENCE OF A MAXIMAL SOLUTION FOR QUASIMONOTONE ELLIPTIC SYSTEMS

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Dedicated to the memory of Peter Hess

Abstract. Suppose the quasimonotone elliptic system has a supersolution above a subsolution. Then it is shown that there exists a maximal solution in between. Solutions and sub/supersolutions are defined in the spaces $C(\bar{\Omega})^n$ and $W_1^2(\bar{\Omega})^n$. The regularity assumptions for the equations are optimal: for fixed off diagonal terms the coupling functions are Carathéodory; for fixed diagonal and space variable these functions are increasing and not necessarily continuous. The basic ingredients are a version of the maximum principle, the Schauder fixed point theorem for the C-case and an existence theorem of J.L. Lions for the W-case.

1. Introduction. In this paper we consider systems of elliptic equations of the form

$$\begin{cases} -\Delta\vec{u} = F(\vec{u}) & \text{in } \Omega, \\ \vec{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^m$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ is a given function.

Problem (1) has been studied extensively in the literature. See e.g. [4], [10], [15], [30] or [18] and the references therein. For its connection with population dynamics and combustion theory, see [7], [9], [14], and [8].

Here we shall study system (1) by using a consequence of the maximum principle, namely the 'sub-supersolution method' (see [4]). It is well known ([27]) that the maximum principle in its various forms does not hold in the framework of vector valued functions (e.g. systems of equations) if one does not assume some structural condition on the coupling. For weakly coupled systems the structural condition is cooperativity (linear coupling) or quasimonotonicity (nonlinear coupling). If one allows coupling also in the derivatives then the maximum principle does not hold even for very simple systems ([27]). Hence in that case there is in general no hope to obtain existence theorems based on the maximum principle ([29]). The above reason

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