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A KORN'S INEQUALITY FOR FUNCTIONS WITH DEFORMATION IN $\mathbf{L^1}(\mathbb{R}^2)$ AND $\mathbf{L^1}(B^2,S^1)$

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1. Introduction. The aim of this paper is to prove that some kind of Korn's inequality, which is false in the space $LD(\mathbb{R}^N, \mathbb{R}^N)$, whose definition is recalled below, can be true for functions defined on the two-dimensional unit ball B^2, with values in a one dimensional variety. More precisely we shall prove here that if $u \in L^1(B^2, S^1)$ (B^2 is the unit ball of $\mathbb{R}^2$ and S^1 is the one dimensional sphere), and if u has its symmetric derivatives in L^1, ($\varepsilon_{11}(u) = u_{1,1}$, $\varepsilon_{22}(u) = u_{2,2}$, $\varepsilon_{12}(u) = (u_{1,2} + u_{2,1})/2$). Then u is in fact in $W^{1,1}$ and the following equality holds true, almost everywhere

$$|\nabla u|^2(x) = 4\left(\varepsilon_{12}(u)\right)^2(x) + \left(\varepsilon_{22} - \varepsilon_{11}\right)^2(u)(x). \tag{1.1}$$

Let us recall first a few facts about the space $LD(\Omega, \mathbb{R}^k)$ and about Korn's inequality in $W^{1,p}$ for $1 < p < \infty$.

Let Ω be a bounded open set of $\mathbb{R}^N$ whose boundary is Lipschitzian; the space of functions with deformations in L^1 is defined as (cf. [15], [16]):

$$LD(\Omega, \mathbb{R}^N) = \Big\{ u \in L^1(\Omega, \mathbb{R}^N),\ \varepsilon_{ij}(u) = \frac{u_{i,j} + u_{j,i}}{2} \in L^1,\ \forall (i,j) \in [1, N]^2 \Big\}. \tag{1.2}$$

It is not difficult to see that

1) LD is a Banach space for the norm

$$|u|_1 + \sum_{1 \le i,j \le N} \left|\varepsilon_{ij}(u)\right|_1,$$

where $|\cdot|_1$ denotes the L^1 norm on Ω.

2) By a classical regularization process, one can prove that $\mathcal{C}^\infty(\Omega, \mathbb{R}^N)$ is dense in $LD(\Omega, \mathbb{R}^N)$ for the norm defined in 1).
3) On every $\mathcal{C}^1$−submanifold Σ of dimension $N - 1$, the trace of u on Σ is well defined, and belongs to $L^1(\Sigma)$. Moreover, the trace map is continuous and possesses a continuous inverse.
4) Korn's inequality is not true in $W^{1,1}$: in other words, this means that there exist functions which have deformation in L^1, such that ∇u does not belong to L^1. This may be proved by using a construction of D. Ornstein [13], which exhibited a sequence u_n of non trivial functions in $\mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |u_{n,12}| \ge n \Big\{ \int_{\mathbb{R}^N} \left(|u_{n,11}| + |u_{n,22}|\right) \Big\}.$$

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