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NONHOMOGENEOUS TIME–DEPENDENT NAVIER–STOKES PROBLEMS IN L_p SOBOLEV SPACES

Gerd Grubb

Copenhagen University, Mathematics Department, Universitetsparken 5, DK-2100 Copenhagen, Denmark

(Submitted by: Y. Giga)

Introduction. The time-dependent Navier-Stokes system of equations (see (1) below) describes the motion of a viscous fluid in a domain Ω. The problem is nonlinear; and in the incompressible case that we consider here, the linearized version (the Stokes problem) is degenerate parabolic. Problems with a homogeneous Dirichlet condition $u|_{\partial\Omega} = 0$ have been most frequently studied; but also other boundary conditions (of Neumann type or intermediate types) are of interest, e.g. for free boundary problems, and so are nonhomogeneous boundary conditions. There are existence theorems for time-global weak solutions, but as the uniqueness and regularity questions are not yet fully clarified (in dimension $n \geq 3$), the construction of strong (possibly local) solutions is of great interest too, and has been worked out in increasing generality. A crucial step is the study of the linearized case. (For a good survey on weak and strong solutions, see, e.g., Giga [4].)

The linearized case presents the difficulty that the problem is not truly parabolic, only degenerately so, whereby standard methods for evolution differential equations are not directly applicable. A main strategy for the case with homogeneous Dirichlet condition has been to reduce the linear problem to the solenoidal space J_0 consisting of functions u with $\operatorname{div} u = 0$ and $\vec{n} \cdot u|_{\partial\Omega} = 0$, where the Stokes operator takes the form of the Laplacian composed with the projection onto J_0; see, e.g., the works of Ladyzhenskaya [18], Fujita-Kato [2], Temam [24] for L_2 spaces, and numerous further developments, e.g. Solonnikov [21,22], Giga and Miyakawa [5], [3], von Wahl [27], for L_p spaces.

However, this strategy does not in a natural way allow nonhomogeneous boundary conditions; and it does not apply in the same way to Neumann problems, where the Stokes operator has a less simple form on the relevant solenoidal space J of functions with $\operatorname{div} u = 0$. We also remark that working in solenoidal spaces puts severe limitations on the usual PDE techniques of changing variables and using cut-off functions, that easily destroy the condition $\operatorname{div} u = 0$.

Motivated by the Neumann problem, the author and V.A. Solonnikov put forward a new method for constructing strong solutions in a series of papers [13–16]. For the degenerate parabolic linearized problem, the method consists of *eliminating the degeneracy,* carrying the problem over to a truly parabolic problem. The price one pays is that a certain non-local term (a so-called singular Green operator) then appears in the equation, so that one has to appeal to the theory of *pseudodifferential* boundary problems, as introduced in Boutet de Monvel [1] for elliptic problems and in Grubb [7] for parabolic evolution problems, and further developed in [14]. But then one has the advantage of drawing on a complete solvability theory in full Sobolev spaces; and in particular one gets a precise analysis of the necessary compatibility conditions. The program was carried out for L_2 Sobolev spaces in [13–16], for Neumann problems as well as for the Dirichlet problem and certain intermediate problems.

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