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ON SOME SPACES OF FUNCTIONS WITH BOUNDED DERIVATIVES BETWEEN MANIFOLDS

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This paper is devoted to some properties of the spaces of distributions between two manifolds M, N, whose certain derivatives are bounded measure. M and N are supposed to be $\mathcal{C}^1$, compact, M being with or without boundary, N being without boundary. Sobolev spaces between manifolds $W^{1,p}(M,N)$ have been studied by Bethuel and Zheng [2] and Escobedo [10] for higher order spaces $W^{m,p}$. By the Theorem of Nash-Moser, N is isometrically imbedded in some space $\mathbb{R}^\ell$ and then $W^{1,p}(M,N) \hookrightarrow W^{1,p}(M,\mathbb{R}^\ell)$. Unfortunately all the properties of $W^{1,p}(M,\mathbb{R}^\ell)$ cannot be extended to $W^{1,p}(M,N)$. For example, density of smooth maps $\mathcal{C}^\infty(M,N)$ does not always hold in $W^{1,p}(M,N)$. For $p \geq \dim M$ it is true, and for $p < \dim M$ it depends on the homotopy group $\pi_{[p]}(N)$. We refer to [1] for complete results concerning that question. In [4], we studied the particular case $p = 1$, and $N = S^1$. We gave a necessary and sufficient condition on $u \in W^{1,p}(M,S^1)$ to be a strong limit in $W^{1,1}$ of maps in $\mathcal{C}^\infty(M,S^1)$. At the same time, We remarked that weak-star density of $\mathcal{C}^\infty(M,S^1)$ holds in $W^{1,1}$.

We define here the spaces $BV(M,S^1)$ and $BD(M,S^1)$ which are naturally related with the theory of calculus of variations and to plasticity.

$$BV(M,S^1) = \{u \in L^1(M,S^1), \quad \nabla u \in M^1\},$$

where M^1 denotes the space of bounded measures on M.

When dim $M = 2$,

$$\begin{aligned} BD(M,S^1) &= \{u \in L^1(M,S^1),\ (\nabla u +^t \nabla u) \in M^1\} \\ &= \{u \in L^1(M,S^1),\ \forall\, i,j \in [1,2],\ \varepsilon_{ij}(u) = \frac{u_{i,j} + u_{j,i}}{2} \in M^1\}. \end{aligned}$$

And the question we raise here is

Is $\mathcal{C}^\infty(M,S^1)$ weakly dense in $BV(M,S^1)$ and $BD(M,S^1)$?

We prove the weak density result of $\mathcal{C}^\infty(M,S^1)$ in $BV(M,S^1)$ in the second section. Let us note that for $k \geq 2$, $\mathcal{C}^\infty(M,S^k)$ is weakly dense in $BV(M,S^k)$ by standard

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