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SOME ERGODIC PROBLEMS FOR HAMILTON–JACOBI EQUATIONS IN HILBERT SPACE

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1. Introduction. In this paper, we study first-order Hamilton-Jacobi equations in a bounded smooth domain Ω of a Hilbert space X, with the Neumann boundary condition

$$H(x, \nabla u_\lambda(x)) + \lambda u_\lambda(x) - f(x) = 0 \qquad \text{in} \quad \Omega, \tag{1}$$

$$\langle \nabla u_\lambda(x), n(x)\rangle = 0 \qquad \text{on} \quad \partial\Omega, \tag{2}$$

where λ is a positive number, $u_\lambda(x)$ is a scalar function on $\overline{\Omega}$, ∇u_λ denotes the Fréchet derivative of u_λ, H is a given continuous function on $\overline{\Omega} \times X$—called the Hamiltonian—and f is a continuous function in $\overline{\Omega}$.

Our aim is to analyze the convergence of the term $\lambda u_\lambda(x)$ as $\lambda \to 0$; in fact, for some class of the Hamiltonians H, $\lambda u_\lambda(x)$ converges to a unique value d which depends on H and f. Such results were obtained by P.L. Lions in [6] when Ω is a bounded domain in $\mathbb{R}^N$ by the viscosity solutions theory, under the following condition on H

$$H(x,p) \nearrow \infty \qquad \text{as} \quad |p| \to \infty \quad \text{uniformly in} \quad x \in \overline{\Omega}. \tag{3}$$

The study of the convergence of $\lambda u_\lambda \to d$ (for each f) is very much related to the so-called ergodic problems (long time average control problems; see [1], [6]). In view of the recent research concerning Hamilton-Jacobi equations in infinite dimensional spaces, initiated by M.G. Crandall and P.L. Lions (see [3]), a natural question is then to extend the results in [6] to our case.

Thus, in the following, we shall establish the uniqueness and the existence of (1)–(2), by the Perron's method which is available in the infinite dimensional space (see Ishii [5]; in which he treated the case $\Omega = X$). After that, we will show the convergence of $\lambda u_\lambda(x)$ to a unique number d, under the same assumption (3) as in [6]. So, roughly speaking, the system corresponding to H has the "ergodic property". However, what is different from the case of the finite dimensional space is that, in spite of the uniform Lipschitz continuity of u_λ (i.e., $\exists L > 0$ such that $|\nabla u_\lambda| < L$ for any $\lambda \in (0,1]$), we do not know if $v_\lambda = u_\lambda - u_\lambda(x_0)$ (x_0 is an arbitrary fixed point in Ω) converges uniformly to a function or not, because of the

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