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THE INITIAL-VALUE PROBLEM FOR THE GENERALIZED BURGERS' EQUATION

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1. Introduction. We will study the well- and ill-posedness of the initial-value problem

$$\begin{cases} \partial_t u = \partial_x^2 u - \partial_x(u^{k+1}) \quad x,t \in \mathbb{R}, \quad k = 1,2,3,\dots, \\ u(x,0) = u_0(x). \end{cases} \tag{1.1}$$

We assume the initial data to be in the homogeneous Sobolev space

$$\dot{L}^{p,s}(\mathbb{R}) = \{f \in S'(\mathbb{R}) : \|f\|^*_{p,s} = \|D^s f\|_{L^p_x} < \infty\}, \tag{1.2}$$

where $S'(\mathbb{R})$ is the space of tempered distributions, $D = (-\partial_x^2)^{\frac{1}{2}}$, $1 < p < \infty$ and $s \in \mathbb{R}$. We will also work with the inhomogeneous Sobolev space $L^{p,s}(\mathbb{R})$ defined as above with the norm replaced by $\|f\|_{p,s} = \|(1-D^2)^{\frac{s}{2}} f\|_{L^p_x}$.

This work is motivated by D. Dix's recent paper ([1]). In it he establishes local existence and uniqueness for the initial-value problem (1.1) for the case $k = 1$; that is, Burgers' equation, in the class $C([0,T), H^s)$ for $s > -\frac{1}{2}$. Further, he proves nonuniqueness for $s < -\frac{1}{2}$; hence the previous result is optimal.

Our results include and extend the well-posedness result proved by D. Dix to the case $k \geq 1$ and $1 \leq p < \infty$. In particular we consider the case $p = 2$, $k = 1$ and $s = -\frac{1}{2}$ which is the limiting case not included in D. Dix's paper. (We extend his nonuniqueness result to the class $C((0,T), L^{p,\rho})$, with $p \geq 2$, $\rho < \frac{1}{p} - 1$ and $k = 1$.)

The following scaling argument motivated our work. Let u be a solution of the initial-value problem (1.1) with initial value u_0. Let

$$u_\lambda(x,t) = \lambda^{\frac{1}{k}} u(\lambda x, \lambda^2 t). \tag{1.3}$$

Then $u_\lambda(x,t)$ is a solution of equation (1.1) with initial data

$$u_{\lambda 0}(x) = \lambda^{\frac{1}{k}} u_0(\lambda x). \tag{1.4}$$

The space $\dot{L}^{p,s_{p,k}}(\mathbb{R})$ with $s_{p,k} = \frac{1}{p} - \frac{1}{k}$ leaves the norm of $u_{\lambda 0}$ invariant. Thus it seems natural to study local well-posedness in $\dot{L}^{p,s}(\mathbb{R})$ with $s = s_{p,k}$, which is the critical value. Notice that when $k = 1$ and $p = 2$ we have that $s_{p,k} = -\frac{1}{2}$.

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