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ON STRONG SOLUTIONS OF QUASI-VARIATIONAL INEQUALITIES

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Abstract. Let S be an upper semicontinuous set-valued mapping of a Hilbert space H into the closed convex subsets of H and let φ be a lower semicontinuous mapping of $\mathcal{H} \times H$ into R with $\varphi(p,\cdot)$ being a convex function of H into R and $D(\varphi) = \mathcal{H} \times V$ where V is a Hilbert space. The existence of a solution of the quasi-variational inequality $u \in S(u)$ and $(Au - f, v - u) \geq 0$ for all $v \in S(u)$ with $Au \in \partial_u \varphi(p,u)|_{p=Lu}$ is shown. Applications to elliptic boundary-value problems and to an equilibrium noncooperative constrained n-person game are given.

0. Let H be a real Hilbert space and let S be an upper semicontinuous set-valued mapping of H into the closed convex subsets of H with convex domain $D(S)$. Let φ be a lower semicontinuous mapping of $\mathcal{H} \times H$ into R^+ with $\varphi(p,\cdot)$ being a convex function of H into R^+ and $D(\varphi) = \mathcal{H} \times V$ where V is a Hilbert space dense in H. There exists a bounded linear mapping L of V into the Hilbert space $\mathcal{H}$.

It is the purpose of this paper to study the quasi-variational inequality

$$u \in S(u) \quad \text{and} \quad (Au - f(u), v - u) \geq 0 \tag{0.1}$$

for all $v \in S(u)$ with $Au \in \partial_u \varphi(p,u)|_{p=Lu}$ and $f(u) \in \mathcal{F}(u)$. In (0.1), $\mathcal{F}(u)$ is a set-valued upper semicontinuous mapping of V into the closed convex subsets of H.

Quasi-variational inequalities arise in the study of optimal control problems with impulses, in free boundary problems, in mathematical economics, and were introduced by A. Bensoussan and J.L. Lions ([4]).

In Section 2 the existence of a solution u of (0.1) with Au in H is established, thereby extending an earlier result of Joly and Mosco (cf. [1], page 537) where $\varphi(p,u) = \varphi(u)$ and $\partial\varphi$ is a set-valued mapping of V into its dual V^*. In the applications, the strong solution u of (0.1) given in Section 2 allows us to deduce some global regularity properties for u.

Let g, h be two proper lower semicontinuous convex functions from H and from a Hilbert space U respectively into R^+. Following Barbu, Neittaanmaki and Niemisto ([3]), we consider in Section 3 the optimal control problem

$$\begin{aligned} \inf\big\{ g(y) + h(u) : y \in S(y),\ (Ay - f(y) - Bu, x - y) \geq 0, \\ \{x, u\} \in S(y) \times U,\ Ay \in \partial_y \varphi(p,y)|_{p=Ly},\ f(y) \in \mathcal{F}(y) \big\}. \end{aligned} \tag{0.2}$$

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