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ON THE OPTIMAL LOCAL REGULARITY FOR GAUGE FIELD THEORIES*

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In [5], [6] we have proved that the Maxwell-Klein-Gordon equations, respectively Yang–Mills equations, in $\mathbb{R}^{3+1}$ are well posed in the energy norm. The result relied on the precise structure of the equations relative to the Coulomb gauge, and on the space-time estimates for null forms proved earlier in [4]. However, the results proved in [5], [6] are not sharp; based on the scaling properties of the equation we expect that the Maxwell-Klein-Gordon and Yang-Mills equations in $\mathbb{R}^{n+1}$ are well posed in the Sobolev spaces $H^s(\mathbb{R}^n)$ for all $s > \frac{n-2}{2}$, or even $s \geq \frac{n-2}{2}$ in a suitable sense, at least when $n \geq 4$. In particular, for the critical case $n = 4$ the optimal exponent $s = 1$ corresponds to the energy norm. In this case local well-posedness for finite energy solutions would also imply global well-posedness. In this paper we present the main estimates which allow one to prove local well posedness in H^s for all $s > \frac{n-2}{2}$ in the case of the M.K.G. equations in $4+1$ dimensions. For simplicity we will demonstrate the relevance of our estimates for a model problem which presents the main difficulties manifest in the Maxwell-Klein-Gordon expressed relative to the Coulomb gauge.

We consider systems of wave equations in $\mathbb{R}^{n+1}$ with quadratic nonlinearities in $\phi = (\phi^I)_{I=1,\dots,M}$, $\psi = (\psi^I)_{I=1,\dots,N}$ written symbolically in the form,

$$\Box\psi = |D|^{-1}Q(\phi,\phi) \tag{1.1a}$$

$$\Box\phi = Q(|D|^{-1}\psi,\phi) \tag{1.1b}$$

subject to the initial conditions, at $t = 0$,

$$\phi = f_0,\ \partial_t\phi = f_1 \quad\text{and}\quad \psi = g_0,\ \partial_t\psi = g_1. \tag{1.1c}$$

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