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SOME EXTREMAL SINGULAR SOLUTIONS OF A NONLINEAR ELLIPTIC EQUATION

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1. Introduction. Let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded domain, and let f be a smooth function on Ω, $f \geq 0$, $f \not\equiv 0$. Let $p > 1$ and consider the semilinear elliptic equation

$$(P_t) \begin{cases} -\Delta u &= u^p + tf \quad \text{in } \Omega \\ u &> 0 \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

where $t \geq 0$ is a parameter. We are concerned with weak solutions of (P_t), and we use the definition introduced in [2]: a weak solution of (P_t) is a function $u \in L^1(\Omega)$, $u \geq 0$, such that $u^p \delta \in L^1(\Omega)$, where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$, and such that

$$-\int_\Omega u \Delta\zeta \, dx = \int_\Omega (u^p + tf)\zeta \, dx$$

for all $\zeta \in C^2(\bar{\Omega})$ with $\zeta = 0$ on $\partial\Omega$. We start by mentioning some well-known facts (see for example [2], [1], [6]).

Theorem 1. *There exists $0 < t^* < \infty$ such that for $0 < t < t^*$, (P_t) has a unique minimal solution $\underline{u}(\cdot, t)$ (which is smooth), for $t = t^*$ (P_{t^*}) has a unique solution u^* (possibly unbounded), and for $t > t^*$ there is no solution of (P_t) (even in the weak sense). Moreover, $\underline{u}(\cdot, t)$ depends smoothly on $t \in (0, t^*)$, increases as t increases, and $\underline{u}(\cdot, t) \nearrow u^*$ almost everywhere in Ω, as $t \nearrow t^*$.*

We call u^* the extremal solution. An important feature of the minimal solution $\underline{u}$ is that the linearized operator at $\underline{u}$, $-\Delta - p\underline{u}^{p-1}$ has a positive

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