Differential and Integral Equations Volume 16, Number 1, January 2003, Pages 1–12

REMARKS ON THE STRONG MAXIMUM PRINCIPLE

Haïm Brezis and Augusto C. Ponce
Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, B.C. 187
4 Pl. Jussieu, 75252 Paris Cedex 05, France
and
Rutgers University, Dept. of Math., Hill Center, Busch Campus
110 Frelinghuysen Rd, Piscataway, NJ 08854

1. Introduction

The strong maximum principle asserts that if u is smooth, $u \geq 0$ and $-\Delta u \geq 0$ in a connected domain $\Omega \subset \mathbb{R}^N$, then either $u \equiv 0$ or $u > 0$ in Ω. The same conclusion holds when $-\Delta$ is replaced by $-\Delta + a(x)$ with $a \in L^p(\Omega)$, $p > \frac{N}{2}$ (this is a consequence of Harnack's inequality; see e.g. [6], and also [7, Corollary 5.3]. Another formulation of the same fact says that if $u(x_0) = 0$ for some point $x_0 \in \Omega$, then $u \equiv 0$ in Ω. A similar conclusion fails, however, when $a \notin L^p(\Omega)$, for any $p > \frac{N}{2}$. For instance, $u(x) = |x|^2$ satisfies $-\Delta u + a(x)u = 0$ in B_1 with $a = \frac{2N}{|x|^2} \notin L^{N/2}(\Omega)$.

If u vanishes on a larger set, one may still hope to conclude, under some weaker condition on a, that $u \equiv 0$ in Ω. Such a result was obtained by Bénilan-Brezis [2, Appendix D] (with a contribution by R. Jensen) in the case where $a \in L^1(\Omega)$ and $\operatorname{supp} u$ is a compact subset of Ω. Their maximum principle has been further extended by Ancona [1], who proved Theorem 1 below.

We recall that a function $v : \Omega \to \mathbb{R}$ is quasicontinuous if there exists a sequence of open subsets (ω_n) of Ω such that $v|_{\Omega \setminus \omega_n}$ is continuous $\forall n \geq 1$ and $\operatorname{cap} \omega_n \to 0$ as $n \to \infty$, where $\operatorname{cap} \omega_n$ denotes the H^1-capacity of ω_n.

Theorem 1 ([1])**.** *Assume $\Omega \subset \mathbb{R}^N$ is an open bounded set. Let $u \in L^1(\Omega)$, $u \geq 0$ a.e. in Ω, be such that Δu is a Radon measure on Ω. Then there exists $\tilde{u} : \Omega \to \mathbb{R}$ quasicontinuous such that $u = \tilde{u}$ a.e. in Ω.*

Let $a \in L^1(\Omega)$, $a \geq 0$ a.e. in Ω. If

$$-\Delta u + au \geq 0 \quad \textit{in } \Omega,$$

Accepted for publication: June 2002.
AMS Subject Classifications: 35B05, 35B50.