COMMUN. MATH. SCI.
Vol. 8, No. 4, pp. 999–1020

STRONG CONVERGENCE OF PRINCIPLE OF AVERAGING FOR MULTISCALE STOCHASTIC DYNAMICAL SYSTEMS*

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Abstract. In this paper, we study stochastic differential equations with two well-separated time scales. We prove that the rate of strong convergence to the averaged effective dynamics is of order $O(\varepsilon^{1/2})$, where $\varepsilon \ll 1$ is the parameter measuring the disparity of the time scales in the system. The convergence rate is shown to be optimal through examples.

Key words. Stochastic Differential Equations, Time Scale Separation, Averaging of Perturbations.

AMS subject classifications. 60H10, 60H10, 70K65.

1. Introduction

Consider the following stochastic dynamical system with a time scale separation measured by $\varepsilon \ll 1$:

$$\begin{aligned} \dot{X}_t^\varepsilon &= a\big(X_t^\varepsilon, Y_t^\varepsilon, \varepsilon\big) + \sigma\big(X_t^\varepsilon, Y_t^\varepsilon, \varepsilon\big)\dot{W}_t, & X_0^\varepsilon &= x, \\ \dot{Y}_t^\varepsilon &= \frac{1}{\varepsilon} B\big(X_t^\varepsilon, Y_t^\varepsilon, \varepsilon\big) + \frac{1}{\sqrt{\varepsilon}} C\big(X_t^\varepsilon, Y_t^\varepsilon, \varepsilon\big)\dot{W}_t, & Y_0^\varepsilon &= y, \end{aligned} \tag{1.1}$$

where $X_t^\varepsilon \in \mathbb{R}^n$, and $Y_t^\varepsilon \in \mathbb{R}^m$ are variables in vector spaces and W_t is a standard d-dimensional Wiener process. $a(\cdot) \in \mathbb{R}^n$, $B(\cdot) \in \mathbb{R}^m$, $\sigma(\cdot) \in \mathbb{R}^n \times \mathbb{R}^d$ and $C(\cdot) \in \mathbb{R}^m \times \mathbb{R}^d$ are all functions of $O(1)$ magnitude. Systems in the form of (1.1) arise from a wide range of applications including chemical kinetics, material sciences, fluid dynamics, and finance. We have assumed that the phase space can be decomposed into slow degrees of freedom x and fast degrees of freedom y. Under appropriate assumptions on $B(\cdot)$ and $C(\cdot)$, the dynamics for Y_t^ε with $X_t^\varepsilon = x$ fixed is ergodic with a unique invariant measure $\mu_x^\varepsilon(dy)$. In this case, the Principle of Averaging has been proved such that in the limit of $\varepsilon \to 0$, X_s^ε converges to a stochastic differential equation of the following form:

$$\dot{\bar{X}}_t = \bar{a}\big(\bar{X}_t\big) + \bar{\sigma}\big(\bar{X}_t\big)\dot{W}_t, \qquad \bar{X}_0 = x, \tag{1.2}$$

where

$$\begin{aligned} \bar{a}(x) &= \lim_{\varepsilon \to 0} \int a(x,y,\varepsilon)\mu_x^\varepsilon(dy), \\ \bar{\sigma}(x)\bar{\sigma}^T(x) &= \lim_{\varepsilon \to 0} \int \sigma(x,y,\varepsilon)\sigma^T(x,y,\varepsilon)\mu_x^\varepsilon(dy). \end{aligned} \tag{1.3}$$

From the point of view of numerical analysis, an important question is in which sense, as well as how fast, the system will converge to the effective dynamics. The recent motivation for this problem is the progress on numerical methods for dynamical systems with multiple time scales. In [12], a multiscale integration scheme was

*Received: November 13, 2008; accepted (in revised version): February 17, 2010. Communicated by Jonathan Mattingly.
Supported by NSF-DMS 0609315.
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