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ON THE ENERGY CONSERVATION BY WEAK SOLUTIONS OF THE RELATIVISTIC VLASOV-MAXWELL SYSTEM*

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Abstract. We show that weak solutions of the relativistic Vlasov-Maxwell system preserve the total energy provided that the electromagnetic field is locally of bounded variation and, for any $\lambda > 0$, the one-particle distribution function has a square integrable λ-moment in the momentum variable.

Key words. Vlasov-Maxwell, weak solutions, conservation of the total energy.

AMS subject classifications. 35Q61, 35Q83.

1. Introduction

Consider an ensemble of relativistic charged particles that interact through their self-induced electromagnetic field. If collisions among the particles are so improbable that they can be neglected, then the ensemble can be modeled by the so-called relativistic Vlasov-Maxwell (RVM) system. At any given time $t \in]0,\infty[$, the RVM system is characterized by the one-particle distribution function $f = f(t,x,p)$ with position $x \in \mathbb{R}^3$ and momentum $p \in \mathbb{R}^3$. The self-induced electric and magnetic fields are denoted by $E = E(t,x)$ and $B = B(t,x)$, respectively. Setting all physical constants to one, the model equations for a single particle species read

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + (E + v \times B) \cdot \nabla_p f = 0 \tag{1.1}$$

$$\frac{\partial E}{\partial t} - \nabla \times B = -4\pi j \tag{1.2}$$

$$\frac{\partial B}{\partial t} + \nabla \times E = 0 \tag{1.3}$$

$$\nabla \cdot E = 4\pi\rho, \quad \nabla \cdot B = 0, \tag{1.4}$$

where $v := p\left(1+|p|^2\right)^{-1/2}$ denotes the relativistic velocity. The coupling of the Vlasov (1.1) and Maxwell equations (1.2)–(1.4) is through the charge and current densities, which we denote by $\rho = \rho(t,x)$ and $j = j(t,x)$ respectively. They are defined by

$$\rho := \int_{\mathbb{R}^3} f \, dp, \quad j := \int_{\mathbb{R}^3} v f \, dp. \tag{1.5}$$

We define the Cauchy problem for the RVM system by (1.1)–(1.5) with initial data

$$f_{|t=0} = f_0, \quad E_{|t=0} = E_0, \quad B_{|t=0} = B_0, \tag{1.6}$$

satisfying (1.4) in the sense of distribution. It is not difficult to check that if (1.4) holds at $t = 0$, then it will do so for all time in which the solution exists. Thus, equations (1.4) can be understood as a mere constraint on the initial data.

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