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STEADY STATES OF THE 1D DOI-ONSAGER MODEL IN THE STRONG SHEAR FLOW*

GERALD WARNECKE† AND HUI ZHANG‡

Abstract. Here we will investigate the steady states of the Doi-Onsager model in general shear flows. Generally we know that the equilibrium is not the Maxwell-Boltzmann distribution in the shear flow. Here we will give not only the expression of the equilibrium for the 1D Doi-Onsager model in shear flows by rigorous proof, but also a discussion of the critical points of the intensity α and the shear rate γ by numerical calculation. The equilibrium of the particles depends on the competition between the parameters α and γ. When α crosses the first critical point $\alpha_1 \approx 4.083$ from a smaller value, the number of steady states will change from one to three through two. At the same time the shear rate γ depends on α. When α crosses the second critical point $\alpha_2 \approx 5.125$ from smaller values, the number of steady states will change from three to one through two for some related γ. It can be found that some of the equilibria are stable with respect to the energy functional. These results are partly consistent with the work of Marrucci and Maffettone in [20], in which formal analysis was given.

Key words. Steady states, strong shear flow, Doi-Onsager model, critical points.

AMS subject classifications. 82B27, 35Q35.

1. Introduction

It is well-known that the Doi model can describe the dynamical states in dilute polymeric fluids. We consider the behaviour of chain molecules or rod particles. Important effects were found in some numerical simulations, e.g., [9, 10, 11, 14, 19, 20], and theoretical analysis, e.g., [1, 2, 3, 26]. In the simplest case of no flow (velocity $u=0$), it is also called Doi-Onsager model [17, 23, 25]. It can describe phase transition phenomena though it is very simple. Therefore, this model has recently attracted considerable attention [4, 5, 6, 14], especially in the mathematics community [1, 2, 3, 7, 8, 12, 13, 16, 17, 18, 23, 24]. At first, Onsager [22] showed in 1949 that the steady state is the prolate phase when the intensity among the particles is strong. The prolate phase consists of aligned particles. He derived this result via a variational approach under the assumption that the equilibrium was axially symmetric and possessed some explicit expression for the Onsager interaction potential between molecules. This name was attributed to the potential by later authors. In 1959 Maier and Saupe [21] investigated the nematic phase for a simpler potential among the molecules, which is now called Maier-Saupe potential, e.g., [4, 5]. In 2005, a rigorous proof of the Onsager assumption for the Maier-Saupe potential was given in [17, 8, 23]. Meanwhile explicit expressions for all equilibria were shown in [3, 17]. The phase transition phenomenon was proved in the investigation of stability of these equilibria [24]. In [24], many steady states were found in the weak flow for the Doi model, such as flow-aligning, tumbling, log-rolling, and kayaking. Concerning the asymptotic states and the properties of the Doi-Onsager model without flow, Constantin et al. [1, 2] gave many very interesting

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†Institute for Analysis and Numerics, Otto-von-Guericke-University Magdeburg, Universitätsplatz 2, D-39106 Magdeburg, Germany (warnecke@ovgu.de).
‡Corresponding author. School of Mathematical Sciences, Beijing Normal University, Key Laboratory of Mathematics and Complex Systems, MCE, Beijing, 100875, P.R. China (hzhang@bnu.edu.cn). This work is partially supported by an Alexander von Humboldt Fellowship at Otto-von-Guericke-University Magdeburg, the key basic research project of the Ministry of Chinese Education 107016 and the state key basic research project of China 2005CB321704.