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THE EXACT EVOLUTION OF THE SCALAR VARIANCE IN PIPE AND CHANNEL FLOW*

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Dedicated to the sixtieth birthday of Professor Andrew Majda

Abstract. In 1953 G.I. Taylor showed theoretically and experimentally that a passive tracer diffusing in the presence of laminar pipe flow would experience an enhanced diffusion in the longitudinal direction beyond the bare molecular diffusivity, κ, in the amount $\frac{a^2U^2}{192\kappa}$, where a is the pipe radius and U is the maximum fluid velocity. This behavior is predicted to arise after a transient timescale $\frac{a^2}{\kappa}$, the diffusive timescale for the tracer to cross the pipe. Typically, κ is very small, so provided a fairly long time has passed, this is a very large diffusive boost. Before this timescale, the evolution is expected to be anomalous, meaning the scalar variance does not grow linearly in time. A few attempts to compute this anomalous growth have been made in the literature for different special cases with different approximations. Here, we derive an exact approach which provides the scalar variance evolution valid for all times for channel and pipe flow for the case of vanishing Neumann boundary conditions. We show how this formula limits to the Taylor regime, and rigorously study the anomalous regime for a range of initial data. We find that the anomalous timescales and exponents depend strongly upon the form of the data. For initial data whose transverse variation is a delta function on the centerline, the anomalous regime emerges after a timescale, $(\frac{a^4}{\kappa U^2})^{\frac{1}{3}}$, with variance growing as t^{α}, with $\alpha = 4$. In contrast, for the case of uniform data (independent of the transverse variable), the anomalous timescale is $\frac{\kappa}{U^2}$, with exponent $\alpha = 2$, and this result is generalized for generic shear flows given that the initial condition is not a transverse Dirac delta function. Further, these exact formulas explicitly show what features the short time approximations which ignore physical boundaries are able to capture.

Key words. Taylor dispersion, mixing, transport, stochastic and partial differential equations, multi-scale asymptotics, pipe and channel flow.

AMS subject classifications. 82C70, 82C31, 34E13.

1. Introduction

The enhanced diffusion of a passive scalar is a fundamental problem with a long history, dating back to the pioneering work of G.I. Taylor [18] who developed the first theory and experiments for pipe flow, for which the phrase Taylor dispersion was born. Taylor dispersion is the phenomena by which a shear flow boosts the longitudinal diffusivity well above the bare molecular diffusivity, κ; on long times Taylor theoretically predicted and experimentally validated this diffusivity to be $\kappa(1+\frac{U^2a^2}{192\kappa^2})$ for the case of laminar flow in a pipe, where U is the maximum velocity and a is the pipe radius.

Since Taylor, there has been an intense effort to calculate, the general enhanced diffusion coefficient for a given fluid flow, which is known to be a complicated function of the fluid flow structure as well as of the Peclet number, $\mathrm{Pe} = \frac{Ua}{\kappa}$, where U and a are typical flow and length scales. Much of this effort has been explored using the multi-scale asymptotic method of homogenized averaging theory, just one of the many tools Andy Majda has employed in his many studies in this area of his influential research

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